

RB Bates

THE
DOCTRINE
OF
FLUXIONS:

Not only Explaining
The ELEMENTS thereof,
But also its
APPLICATION and Use

In the several Parts of MATHEMATICS
and Natural PHILOSOPHY.

The Second Edition, corrected and greatly enlarged.

Ornari res ipsa negat contenta doceri.

L O N D O N :
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The PREFACE.

TO say any thing in Praise of the Method of Fluxions, or of its Dignity and Rank among the Mathematical Sciences, would be as needless as to describe the Excellency of bright Sun-shine above the twinkling Light of the Stars; since any one who is acquainted with the Sciences will allow it to be a Method of Calculation incomparably superior to all other Methods that ever were known or found out; and beyond which nothing further is to be hoped or expected. It lends its Aid and Assistance to all the other Mathematical Sciences, and that in their greatest Wants and Distresses: It opens and discovers to us the Secrets and Recesses of Nature, which have always before been locked up in Obscurity and Darkness. To this all the noble and valuable Discoveries of the last and present Age are entirely owing: And by this Method Sir Isaac Newton, the worthy Inventor, determined and settled the System of the whole visible World.

The Use and Application of FLUXIONS are exceedingly extensive; for Example, in Trigonometry, it teaches the Computation of Sines, Tangents and Secants; in Arithmetic, the Calculation of Logarithms; in Geometry, drawing Tangents to Curves, finding their Curvatures, their Lengths and Quadratures, the Surfaces and Solidities of Bodies; in Mechanics and Philosophy, the Investigation of the Centers of Gravity and Oscillation, the Vibration of Pendulums, the Laws of Centripetal Forces, the Times, Velocities, and Spaces described by Bodies acted upon by any Forces, the Motions and Resistances of Bodies in Mediums, &c. These are some of the numberless Instances, wherein Fluxions are applied with such wonderful Success. And though some few of these may be (and actually have been) hammer'd out with great Labour and Difficulty by other Methods; yet the Process of none of them can in the least be compared with that Beauty, Simplicity, and charming Elegance, with which

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the Method of Fluxions performs all these Things. In short, the Method of Fluxions is capable of resolving such Difficulties as raise the Wonder and Surprise of all Mankind, and which would in vain be attempted by any other Method whatsoever. So that it is justly esteemed the greatest Work of Genius, and the noblest Thought that ever entered the human Mind.

The Method of Fluxions is founded upon this most simple and obvious Principle, viz. that any Quantity may be supposed to be generated by continual Increase, after the same Manner that Space is described by local Motion. The great and noble Inventor tells us, that in this Method he considers Things as generated by continual Increase, after the Manner of a Space which a Thing or Point in Motion describes. Now the Conception of this is exceeding easy and natural; for we every Day see with our own Eyes, all Kinds of Lines and Figures described by the Motion of Bodies: This Principle then will be easily admitted. And further, since we also see by Experience, that these very Lines and Figures are described, some with greater Degrees of Velocity, some with less, some with Motions continually accelerated or retarded, and some with uniform Motions: We shall easily understand, that any one of these Lines or Spaces has in every Point of its Description a certain Degree of Increase determinate in itself, and peculiar to that Point, and which is the same with the Velocity of the Thing that describes it. And to determine this Velocity, or this Degree of Increase, in any given Point of the generated Quantity, is the same Thing as finding the Fluxion of a proposed variable Quantity, and is the Foundation of all the Arithmetick of Fluxions. And to determine this truly is of the greatest Consequence for establishing the Theory.

Many Disputes and Objections have been advanced against the Truth of the Method of Fluxions; and amongst these Disputants, as it commonly happens, those have been the most inveterate, who understood the least of the Matter. To answer all the Cavils that have been offered will be of little Consequence in any thing, and of none at all for settling

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settling the true Notion of Fluxions. Therefore, instead of that, I shall, by following Nature as closely as I can, endeavour to give the unprejudiced Reader a clear and true Idea thereof; and then perhaps he will be able to judge for himself, whether the Principles of this noble Art be capable of Demonstration or not. In order to this, let us assume what has been before laid down, that any generated, flowing Quantity is analogous to a Line described by a moving Point, and that the Velocity of this Point in any Place represents the Fluxion of that Quantity in the correspondent Place of the Fluent; now I shall consider the Generation of this Line, instead of the Fluent, as being more easily understood.

It is the general Practice in Mechanics, to measure the Velocity of a Body by the Space uniformly described in a given Time. For Velocity being that by which a Body is carried through a given Space in a given Time, therefore Velocity must be looked upon as the proper efficient Cause of the Space described; and the Space described the adequate Effect of that Cause. Now suppose a right Line described with any Sort of Velocity, accelerated, or retarded, at Pleasure, and that we would enquire what is the Velocity of it in any given Place. If we take a small Part of the Line, which the moving Point describes just before it arrives at that Place, and call it an Increment, and suppose it to be described in a very small given Time; then this Increment will nearly measure the Velocity of the describing Point at the Place proposed, and is sufficient to give a vulgar Notion of the Degree of Velocity required. Now if this right Line was described uniformly, this would accurately measure the Velocity. But since that Increment is described with a Velocity, by Supposition, continually variable, therefore this Notion we have here obtained is to be corrected; the first Notions we get of any Subject are generally incorrect, and demand a nicer View, and a more accurate and philosophical Examination, before we can acquire Notions that are perfect and adequate. Here then it will be very evident, if we take still a lesser and a lesser Increment, by which the Velocity

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is measured, as the Point still draws nearer the proposed Place; we approach nearer and nearer to a uniform Velocity, till the Difference be less than any assignable: And this Increment will differ from the true Measure of the Velocity, by less than any given Difference: And as this Increment continually diminishes, till at last it vanishes, it approaches continually to that Measure, till the Difference vanishes with it.

Now although by diminishing the Increment at Pleasure we can approach within any Degree of Exactness to the Velocity required, yet since no Increment can be taken so small, but it is still further divisible ad infinitum; and since the Velocity is by Supposition continually variable, it is plain, there can be no two Points of this Increment in both which the Velocity is accurately the same. It is therefore most manifest, that the Velocity here enquired after is peculiar to one only indivisible Point; and that Point is the Place where the Increment ends, or vanishes into nothing. Here then we see plainly, that the Velocity in any given Point of the Line described (or, which is the same Thing, that the Fluxion in any given Point of a generated Quantity) has a certain, fixed, determinate Value, proper to that Point of it alone: And this furnishes the Mind with that accurate abstract Idea, which we ought to form of this Velocity or Fluxion. And here we may observe, that this Degree of Velocity (or Fluxion) we have here been considering, and which continues but a Moment, differs from the same Degree of Velocity (or Fluxion) which continues for any given Time, and by which a given Space is actually described; these, I say, differ no otherwise than as a Cause in POWER differs from a Cause in ACT.

Here a metaphysical Disputant may demand, how it comes to pass, that any Velocity which continues for no Time at all, can possibly describe any Space at all; or whether its Effect be absolutely nothing, or an infinitely small Quantity, or what it is. Here then it is, that our Reason is at a Stand, and the human Faculties are quite confounded, lost, and bewildered. We are puzzled
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and perplexed by endeavouring to examine into the Nature of we do not know what, nor whether it is something or nothing: And at best is some such subtle, fleeting Thing, as the Mind can lay no Hold on, nor form any Idea of. Now whether such subtle Questions will be ever determined, or not, yet there is one Refuge for us, viz. that it is nothing at all to our Purpose what they are: And therefore we may safely leave these deep Speculations to those that have more Business with them. The Method of Fluxions has no Dependence on these mysterious Disquisitions. What I apprehend the Method of Fluxions to be concerned in, is, not what any single abstract Velocity can describe or generate of itself, but what a continual and successively variable Velocity can produce in the whole. And here I think no Reason can be assigned, why a variable Cause should not produce a variable Effect, as well as a permanent Cause a permanent and constant Effect. For since every Effect has a co-instantaneous Existence with its efficient Cause, and is always perfectly connected with it; all the Difference can only be this, that the continual Variation of the Effect must always depend on, and be proportional to, the continual Variation of the Cause that produces it. And this will always be true, though we have no Ideas at all of the perpetually arising Increments, or their Magnitude in their nascent or evanescent State, that have so much, and to so little Purpose, confounded and puzzled the mathematical World. And whether we can or we cannot conceive the formal Nature or Manner of existing of a Thing just arising out of nothing, or beginning to be; or whether a nascent or evanescent Quantity be any thing or nothing; yet the Truth of the Method of Fluxions will stand just as it did. But these sort of Disputes have been artfully introduced for no other Purpose but to involve the Subject in Obscurity, to darken the Reader's Judgment, and thereby to mislead and divert him from pursuing the principal Business in Hand, that is, from considering the proper Evidence on which alone this Doctrine is founded; by insinuating, that the Knowledge
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of these Things is essential to the very Nature and Foundation of Fluxions: When, it is evident, all that the Method of Fluxions does or ever did propose, being either to determine the Velocity (or Fluxion) wherewith a generated Quantity increases in any given Point; or else to sum up all that has been described or generated by such continually variable Fluxion, during any given Time, or to any given Point of the Fluent or generated Quantity. These two Things alone are the two Bases on which this noble Structure (the Method of Fluxions) is to be erected. All metaphysical Speculations, of what Nature soever, having no Business here.

I shall now bring an Instance or two out of the Phenomena of Nature, which will help the Reader's Notions a little; and will shew, that what has been said before, concerning the Nature and Idea of Fluxions, is really true, and agreeable to the Nature and Constitution of Things. Let a heavy Body descend through a perpendicular Height of $16\frac{1}{2}$ Feet in one Second of Time, according to the Gallilean Hypothesis of Gravity; then at the End of this Second of Time, the Body has acquired a Velocity of $32\frac{1}{2}$ Feet in a Second; which therefore is accurately known. Now take any Point *A* in the right Line, at any given Distance from the Place the Body fell from, and the Velocity which the falling Body has in the Point *A* may be most accurately computed. But take any Point above *A*, though at ever so small a Distance, if it be distant at all from *A*, and the Velocity in that Point will always be something less than in the Point *A*. And in like manner the Velocity at any Point below *A*, though indefinitely near it, will be something greater than in *A*; and therefore it is plain, that to the Point *A*, there belongs a certain determined Degree of Velocity, which belongs to no other Point in the whole Line, and this is accurately the Fluxion of that right Line in the Point *A*; and is the Velocity with which the Body would proceed, if the Force of Gravity should be supposed immediately to cease when the Body arrives at *A*, and to act no longer.

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Let there be a glass Tube, open at both Ends, and whose Concavity is of different Diameters in different Places, let it be immersed in a running Stream of clear Water, so that the Water may flow freely through it, and always fill the Tube. Then it is evident, that in different Places of the Tube, the Velocity of the Water will be reciprocally as the Squares of the Diameters of the Tube, in these Places, and will therefore be different. Therefore if you mark any Place in the Side of the Tube, and suppose a Plane to pass through the Tube perpendicular to the Axis, or to the Motion of the Water, then the Water will always pass through this Section with a certain determinate Velocity. But suppose another Section to be drawn, though ever so near the former, then (by reason of the supposed different Diameters) the Water flows through this with a Velocity different from that it did at the former: And therefore that given determinate Velocity belongs only to one single, indivisible Point, or Section of the Tube, and this is the Fluxion of the Space which the Fluid describes at that Section; and is that uniform Velocity with which the Fluid would continue to move, if the Diameter continued the same through the succeeding Part of the Tube. Something like this may be observed in a River, for there the Velocity is greatest, where the Dimensions are least, and less where these are greater.

Again, let a hollow Cylinder be filled with Water, and let it flow freely out through a Hole at the Bottom of it. It is well known, that the Velocity of the effluent Water depends on the Height of the Water within the Cylinder; and therefore, since the Surface of the incumbent Water continually descends without any the least Stop, the Velocity of the effluent Stream will continually decrease, till it all be run out. Therefore it is plain, there can be no two Moments of Time, succeeding each other ever so nearly, wherein the Velocity of the running Water is precisely the same. And therefore the Velocity that the effluent Water has at any given Point of Time, belongs only to that one particular, indivisible Moment of Time,

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and no other: And this is accurately the Fluxion of the Fluid flowing out at that Moment of Time. Now if precisely at that Moment you begin and continue to pour more Water into the Cylinder, so that the Surface of the Water may descend no lower, but keep its Place; then the effluent Water will also retain its Velocity, and continue to be the Fluxion of the Fluid as before. Now these are the genuine Effects and Operations of Nature itself; and do, in a manner visibly, confirm the Truth of what has been said of the Nature of FLUXION.

From these Examples, and many more that might be produced, it is clear to me, that it is an essential Property of the Fluxion of a generated Quantity, that it does not retain any one determined Value for the least Space of Time whatever; but at the Moment it arrives at that Value, the same Moment it leaves it again; so that it only passes gradually and successively through all the indefinite Degrees contained between the two extreme Values which are the Limits thereof, during the Generation of the Fluent: That is, in case the Fluxion be variable at all; but if it is invariable, the extreme Values, and all the intermediate Degrees are but one and the same Value. And therefore, although any determinate Degree of Fluxion does not continue at all; yet every Fluent has (intrinsically) in itself, some determinate Degree of Fluxion, at every determinate, indivisible Moment of Time.

It being now, I suppose, made evident, that every generated Quantity has every where a certain Rate of increasing (called its Fluxion) whose abstract Value is determinate in itself, at any determinate Point of that Quantity: Therefore to find out its Value, or its Ratio to any other Fluxion, is a Problem strictly geometrical. It remains to enquire in general how we must compute this Value. And here the only, or at least the most natural Way is, to get the Proportion of the Increments generated by the Fluxions in all Suppositions of Magnitude of these Increments, and from thence collect the Ratio they first begin with. When the Fluxions and Moments

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Moments of the simple literal Quantities are designed by proper Symbols; it will be easy by the binomial Theorem to find the contemporary Increment of any compound Quantity; for that will be expressed in the Form of a Series. Now although it is evident, that this Increment of the compound Quantity is not accurately the Fluxion of it in any determinate Point, because that very Increment is generated by a Fluxion continually variable; yet it is as evident that it continually approaches to it, by continually diminishing the Increments of the simple Quantities. Here then will be had in general, the Ratio of the Fluxion of a simple Quantity to the Fluxion of that compound Quantity, and in the lowest Terms, and that as near the Truth as we please, whilst we suppose some, though very small, Increment actually described. But since the Ratio of these Fluxions is required for, and belongs only to, some one indivisible Point of the Fluent, that is, in the very Beginning of the Increment, or when there is no Increment at all generated; therefore in this particular Case making the Values of the simple Increments nothing, which before was expressed in general, and then all the Terms wherein they are found will vanish, and what is left will accurately show the Relation of the Fluxions, for that single indivisible Point where the Increment is supposed first to commence, or was required. For this abstract Value of the Fluxion belongs to no more Points than one of the Fluent; and therefore of Consequence the Moments must be made to vanish, after we have seen by the continual Diminution thereof, whither the Ratio tends, and what it continually converges to; which will be as visible to every Body as the very Characters it is written in. And if any one should doubt of the Truth of this, I should for ever despair of convincing him of any Thing at all. The Increments here must necessarily be made use of, not to determine their Magnitude, as some have absurdly imagined, but as a Medium in our Reasoning, to discover the Quantity of the Cause that produces them, they being the continual Effects of the Fluxions; and how can we

judge of the Force or Efficacy of a Cause, without considering the Effect that it does or could produce: For that like Causes are proportional to their Effects can never be denied, except by those that can deny any Thing. This Way of Reasoning and Method of Demonstration then must be exceedingly clear and convincing to all that are duly qualified to examine and consider it, and do not with a most unaccountable Obstinaey, and invincible Prejudice, resolve to yield to no Reason at all, though laid before them as clear as the Sun.

And here it may be worth observing, that in the Process of this Demonstration, the Terms which vanished out of the Increment of the compound Quantity, did plainly arise from, and was generated by the Variation of the Fluxion of that compound Quantity; and the remaining Quantities alone are those generated by the Fluxion itself.

The Easiness and Simplicity of this Method of Demonstration is no small Argument for its Truth and Perfection. The Simplicity of Truth is its great Beauty. And by this Mark it here proves itself to be the genuine Off-spring of Nature and Truth. But if any Persons will not assent to the Truth of these Principles, I would have them suspend their Judgments, lest they make it appear that they have no Judgment at all. In the mean Time let them compare the Results and Conclusions obtained by the Method of Fluxions, with the like Conclusions obtained by other undisputed Principles or Methods of Calculation; and if these Results continually agree, then it is a convincing Proof (at least à posteriori) that the Principles from whence they are deduced must be equally true. But if any Person that plainly discovers himself unacquainted with mathematical Principles, shall, out of his Aversion to these Sciences, cavil and dispute against the Principles here laid down, and which he understands nothing of; and endeavour to put the Issue on such a footing, as neither himself nor any Body else can understand any Thing about it, by running the Account of it into the dark: I think it can be of no Consequenc

sequence at all to trouble a Man's self with this Sort of Anti Mathematicians. Ignorance, Malice, or Prejudice can deserve no Notice.

And thus much I choos'd to say here by Way of Preface, rather than in the Book itself, (which I would not encumber with needless Disputes) concerning the Nature and Principles of Fluxions; a Thing easy enough to be understood, and rendered difficult, more by the intricate Disputes that have been dragged into it by the Enemies of Science, than from the Nature of the Thing itself. The divine Newton (whose Works will last as long as the Sun and Moon) clearly saw that this Matter did not at all require to be built upon any metaphysical Speculations; he, by expressing the simple Moments by general Characters, did thence derive, by infinite Series, the Moments of compound Quantities; from whence he gets the Proportion of the Fluxions for any indetermin'd Values of these Moments, from which general Proportion he at last gains their Proportion for that particular Case where the Moments first begin, or at last vanish into nothing. And thus he has given a Demonstration extremely easy and compleat in it self.

If Arts and Sciences of many hundred Years standing receive daily Improvements and Additions, it cannot be supposed that this most sublime Art of all, found out but Yesterday, can be arriv'd at Perfection all on a sudden. If this Art be so exceedingly useful and valuable, it certainly deserves the Pains and Attention of the learned Mathematicians. And the World must expect, that the Beauty of this Method will excite them to lend all their Assistance towards the Advancement of so noble a Branch of Learning, whether it be in improving the Theory, or facilitating the Practice. Therefore I hope I shall be excus'd at least, if, among others, I endeavour to contribute a little towards this great End.

The following Book is divided into three Sections: In the first are laid down such universal Propositions, as are the Foundation of all that Doctrine. The second Section applies these Principles to the Solution of the most general

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ral Problems, or those of most frequent Use in the Mathematicks: And here many of the Problems so often done by others, are resolved by Methods entirely new, and, I think, more simple; and therefore will by many Persons be more easily apprehended in this Form. And because the Resolution of physical Problems has been little touched on by others, I have added the third Section, where you have the Investigation of some of the chief physical Problems in the Phenomena of Nature. And in this Section, it may perhaps please a Reader of the Principia (the greatest human Production that ever appeared in the World) to see many of the Author's subtle Problems resolved by his own Analysis.

It is not in the least pretended that all Things here treated of are new; for I have collected many Things which I thought material for forming this Doctrine into a regular System; and what was wanting I endeavoured to supply as well as I could; not that I take it to be perfect: For there are many Desiderata still wanting to compleat the Science. Because the Method of finding Fluents by the Tables is exceeding compendious and useful, and has yet been but very slightly passed over by the Writers on this Subject: I have been at the Pains (which was not a little) to compose a new Table, whose Use will appear upon trial to be far more easy and intelligible than any extant; and no less extensive. And for the Explanation and Use of it, I have given a vast Variety of Examples throughout the whole Book. Yet I have not omitted the most general known Rules for finding the Fluents by infinite Series; and have inserted the general Forms of them in the Table. In the Use of which Table there is not the least Difficulty, there being nothing required but a bare Substitution of Quantities. But as to the Resolution of Problems by infinite Series, I have been more sparing of that, because it has been well prosecuted by others. I am not ignorant, that (by the Method of Transmutation of Fluxions) this Table might have been further extended, and other more compounded Forms might have been inserted. But, considering how
seldom

seldom these come in Use, I thought it needless to carry it any further. I have all along, in my Calculations, used distinct Characters for the Fluxions and Moments, since they ought not to be confounded together. And in most Problems (except such where the Reasoning is so obvious as not to need it) I have used both of them; beginning first with the Moments, and substituting at last the Fluxions for the evanescent Moments which are proportional thereto.

In all these Things I have designedly been very short. For the general Rules and Methods of Operation being laid down in as few Words as possible, the Examples will explain their Meaning and Use. In the mathematical Sciences I have taken general Methods to be best; and they that deal in the detail of Things, and spin them out to an unnecessary Length, making thereby a pompous show of Words only, do certainly mispend the Time of their Readers: Since one great End to be aimed at in the Sciences, is to abridge and reduce them to the most general and concise Rules.

As I am not conscious of any Faults I have committed in this Treatise, so I hope they are but few. But in such a vast Variety of Things of the most intricate Nature, it is hardly possible but some will escape. Therefore I must beg of the courteous and good natur'd Reader (for whom alone it was written) that he will rather kindly inform me of my Errors and Defects, than censure and condemn the Book. For as Truth is what I seek, I shall with Pleasure retract or correct any Thing I have written, when it appears inconsistent with that or the Reason of Things.

Lastly, let me acquaint the Reader, that it is indispensably required, that he perfectly understand Arithmetic, Geometry, and Algebra in all their Parts and Improvements, the Methods of Series, Doctrine of Proportions, Nature of Logarithms, Mechanics, and Laws of Motion, &c. all which are to be learned from these particular Sciences to which they belong. For I am clearly persuaded, that it is the best Method to treat every Science

distinct

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distinct and entire by itself, without the Mixture or Interposition of any Thing foreign to the Subject. And therefore in this Treatise I have delivered nothing but the pure Doctrine of Fluxions alone. It would be but lost Labour for any Person unacquainted with these Precognita, to spend any Time in reading this Book; or indeed to attempt to read any such like Treatise with any tolerable Judgment. The Consequence would be, that either the Author or the Difficulty of the Subject must be blamed, as is always the Case; but never the Reader. But then if he comes thus prepared, this will make every Thing easy and pleasant, and he will then find few Difficulties here, but what he will easily surmount. All which I submit to the Perusal of the candid and judicious Reader.

W. Emerson.

P. S. In this second Edition I have made many Alterations and Additions in several Parts of the Book, which are too tedious to mention: But the Additions of principal Note are these, in Sect. I. several more Examples of transforming Fluxions are added to Prop. 9. as also more Examples of finding Fluents, to Prop. 10. There is also some farther Explication of the Use of the Table; likewise Ex. 10 and 11 to Prop. 13, are enlarged, and more Examples added. In Sect. II. Prob. 1. is enlarged, and two more Problems added at the End. Likewise I have added a new Problem at the Beginning of Sect. III. and several more physical Problems at the End of it, with two new Copper Plates.



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E R R A T A.

Note, when b is set to the Page, you must reckon from the Bottom.

Page	Line	Read	Page	Line	Read
6	3	$bqz^m y^m - 2p^2 y^2$	253	1 b.	$\frac{2a}{3a}$
15	9	$x^v = \dot{z}$	260	6	Prop. II.)
12	8	$\dot{z} = x^v y^v - \dot{y}$	11		Vertex
47	8	$-y^2 x x$	15		$\frac{f c^v + g c^w}{n} \times \dot{x}$
125	6	$2R \sqrt{\alpha \beta z^n + \beta^2 z^{2n}}$	262	5	$\frac{\lambda}{a} \frac{a^5 z - a^6}{\lambda}$
	7	$\sqrt{-\alpha} \sqrt{\alpha + \beta z^n}$	263	4 b.	$as : z : +$
		$\sqrt{rr - bb} + b \sqrt{\frac{rr - ss}{ss}}$	264	10	$-\frac{c}{a} u^r$
166	1	Log: $\frac{\sqrt{rr - bb} - b \sqrt{\frac{rr - ss}{ss}}}{ss}$	272	2 b.	$xx = cv ;$
178	2	(add) and Vel. of $w = .414 \times 2b$	278	2	Log: $\frac{dy}{bb} +$
180	6	Log: $X - \text{Log. } x^m$	282	14	$= IR ;$
227	6	$+ m \times m - 1 \times f x^{m-2}$	287	14	(when $s=0, y=a$)
	12	$m = 2, g =$	288	7 b.	Fig. 130.
	14	$- 2ay^3 \times \pi^3$	291	3 b.	near ML,
394		$: 2b \times -2ay^3 - 2a \times 2bx + ab^2 : \times y^3$	1 b.		$mo = Mr,$
239	6 b.	dele (by Form the 16th)	292	8 b.	$= \frac{1}{2} v.$
241	7	$3 - 2d - dd$	294	2	Fig. 148.
401	6 b.	Fig. 59 (in the Margin)	8 b.		dele 148 (Fig.)
246	17	$bc + cd - v$	295	4	Fig. 149
404	248	$\frac{dyy}{2} =$	9		the Curve is
		$BM \times \frac{DR}{2}$	5 b.		that is $Mf -$
406	250	1 b. $BM \times \frac{DR}{2}$	296	12	the Sphere
409	251	8 AbD	299	11	$+ \frac{2mf - ma \times}{2}$
	9	Ab the	12		$\frac{ma}{b} - m^2 \times^2.$
416	16	$DR =$	300	3 b.	$SC - Sd \times d$
421		$z = \frac{1}{m+1} y^{\frac{1}{m} + 1}$	301	10	$= x^u ;$
423	253	1			

E R R A T A.

Page	Line	Read	Page	Line	Read
305	2	$2 = \frac{aay}{2} -$	326	9 b.	$S.mCD$ or
		$\frac{Cr}{CO^2}$	333	15	whence $\frac{\ddot{x}}{xy^2}$
310	8 b.	or as $\frac{Cr}{CO^2}$	334	11	$Br, \ddot{x};$
313	1	Fig. 171.	392	5 b.	$1 - \frac{2}{3}v + 11.23v^2 \times$
315	7 b.	$= \frac{4rzzz}{\sqrt{rr-zz}}$			In Page 265, 266,
316	3 b.	$CQ^2 \times AQ =$			267, 268, 269, read
	ibid	$= x^2v + \frac{1}{3}v^3.$			every where instead
317	13	$4\dot{S}\dot{Z} =$			of e , except Line 5 b
319	9	$c \times 2Ef \times \dot{E}f \times Ef =$			(p.266) read $m - 4p.e;$
326	1	Fig. 201.			and Line 4 b read
	13	Cm is			$4p - 4e - m - n \times eA.$

Corrected

BOOKS printed for J. RICHARDSON, and wrote by
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THE
DOCTRINE
OF
FLUXIONS.

POSTULATUM.

That any Quantity may be supposed to be generated by continual Increase.

HERE Quantities are consider'd, not as composed of an infinite Number of constituent Parts, but as described by a continued Motion. Thus a Line is described by the Motion of a Point; and a Rectangle may be conceiv'd to be generated by the Motion of one Side along the other; and Time proceeds by a regular Flux. And all other Quantities may (by Analogy) be conceived to be generated after the same Manner.

DEFINITIONS.

Definition I.

Quantities generated by a continual Increase are called *Fluents* or *Flowing Quantities*. Those Quantities that always retain the same Value are called *given*, *constant*, *standing*, or *invariable Quantities*; and those

B

that

The DOCTRINE

that are continually changing their Value are called *variable* or *indetermin'd Quantities*.

Thus in a Circle the Diameter is a constant Quantity; and the versed Sine and correspondent Sine are variable Quantities: Also in a Parabola, the Latus Rectum is a constant Quantity, and the Abscissa and correspondent Ordinate are variable Quantities.

Def. II.

The Velocity of the Increase of any generated Quantity, or the Degree of Quickness (or Slowness) wherewith the new Parts of it continually arise, is called its *Fluxion*.

Thus when a Line is generated by the Motion of a Point, the Line itself is the Fluent, and the Velocity of the moving Point is strictly its Fluxion. But Velocity is never properly ascribed to any Thing but local Motion, and is used in this Definition, rather to *describe* what is meant by the Word Fluxion, than to *define* it. Velocity is the same in a *particular* Sense in Relation to the Space described, as Fluxion is in a *general* Sense in Relation to the Fluent generated thereby. Velocity is allow'd by all to be a simple Idea, and so is Fluxion too. When a Man considers the Generation of several Quantities, after this Manner, he will find some to increase faster, others slower; and consequently that there are comparative Velocities (or Fluxions) of Increase during their Generation: And thus he will by Degrees get the Idea of a Fluxion; but without such attentive Consideration, he will never be the wiser for all the Words in the World.

Def. III.

The indefinitely small Portions of the Fluent which are generated in any indefinitely small Portions of Time are called *Moments* or *Increments*. Or if the Fluent decreases, the Portions continually destroy'd are called *Decrements*.

These



These Moments are the immediate Effects of the Fluxions, and are those Quantities by the continual Accession of which the Fluent increases and grows bigger and bigger : That is, any Moment consider'd alone is the adequate Effect of some single determinate Fluxion which is (consider'd as) its generating Cause. Therefore the Moments and Fluxions ought not to be confounded together, since the Moments (being generated by the Fluxions) are as different from the Fluxions, as any Effect is different from its Cause.

Def. IV.

The *Velocity*, *Variation*, or *Quickness* of *Increase* (or *Decrease*) of any Fluxion is called the *second Fluxion*; likewise the *Variation* or *Quickness* of *Increase* of the second Fluxion is called the *third Fluxion*, &c.

As in the Generation of any Fluent, the different Parts of it may be generated faster or slower, that is, its Fluxion at different Times may be unequal; so there must be Degrees of Variation by which it is continually changing, that is, it must have a second Fluxion. And in like Manner this second Fluxion may also be continually variable, and therefore must have a certain Degree of Variation in every Point, or a third Fluxion. And so on.

Def. V.

Contemporary Fluents are those which are supposed to be generated together or in equal Times; or which begin together and end together.

Def. VI.

In any Fluxionary Equation, a *Quantity* of the *first Order* is that which has only one first Fluxion in it; a *Quantity* of the *second Order* has either one second Fluxion or two first Fluxions: *Quantities* of the *third Order*, are third Fluxions, product of three first Fluxions, product of a first and second Fluxion, &c.

The DOCTRINE

NOTATION.

1. The first Letters of the Alphabet, a, b, c , &c. are generally put for standing Quantities; and the last x, y, z , &c. for variable or flowing Quantities.

2. If x or y be put for any fluent or variable Quantity, then the same Letter with a Point over it $\dot{x}, \dot{y}$, denotes the Fluxion of x or y respectively; and the same Letters twice pointed $\ddot{x}$ and $\ddot{y}$, are the Fluxions of $\dot{x}$ and $\dot{y}$, or the second Fluxions of x and y : thus $\dddot{x}$ and $\dddot{y}$ are the third Fluxions of x and y : Likewise $\ddddot{x}$ and $\ddddot{y}$ denote the fourth Fluxions of x and y , &c. also the Fluxion of a or b is 0.

3. Again, $\acute{x}$ denotes the Moment or Increment of x , and $\acute{y}$ the Moment or Increment of y ; likewise $\ddot{x}$ and $\ddot{y}$ denote the Moments of the Moments, or the second Moments of x and y , &c.

4. To the common Algebraic Characters already receiv'd I add this $\propto$, which signifies a general Proportion; thus, $A \propto \frac{BC}{D}$, signifies that A is in a constant Ratio to $\frac{BC}{D}$; that is (if a, b, c, d be other Values of these Quantities) $A : \frac{BC}{D} :: a : \frac{bc}{d}$; and thus every general Proportion is to be understood.

Axiom.

Quantities, which in any finite Time continually converge to Equality, and *before* the End of that Time, approach *nearer* to one another than by any given Difference, do at last become equal.

If any should think this not clear enough to pass for an Axiom, he may consider it thus; let D be their ultimate Difference, therefore they cannot approach nearer to Equality, than by that given Difference D , contrary to the Hypothesis; which Supposition is absurd in all Cases except when D is nothing.

S E C T.

S E C T. I.

The fundamental Principles and Operations of FLUXIONS.

P R O P. I.

The Fluxion of any Fluent or generated Quantity is equal to the Sum of the Fluxions of all the Roots or Sides, each multiply'd continually by the Index of its Power, and by the given Fluent divided by the said Root or Side.

D E M O N S T R A T I O N.

I. **L**ET the Fluent be $b x^m y^n$; now its Fluxion must be a Thing real and determinate in itself, otherwise we are seeking that which has no Existence. By the Notation, $\dot{x}$ and $\dot{y}$ are the Fluxions of x and y ; and will produce Effects, that is, will generate Moments proportional to themselves whilst they retain their Values, which therefore may be expressed by $o\dot{x}$, $o\dot{y}$.

2. Now by the Postulatum, these Moments will increase the Quantities x , y , which therefore will become $x + o\dot{x}$, and $y + o\dot{y}$.

3. Therefore the Fluent $b x^m y^n$ will now become $b \times \overline{x + o\dot{x}}^m \times \overline{y + o\dot{y}}^n = b \times : x^m + m x^{m-1} o\dot{x} + p x^{m-2} o^2 \dot{x}^2, \text{ \&c. } \times : y^n + n y^{n-1} o\dot{y} + q y^{n-2} o^2 \dot{y}^2, \text{ \&c.}$ where p, q are given Quantities.

4. The

4. The last Quantity being actually multiply'd, and $bx^m y^n$ substracted from it; we shall have $bmx^{m-1} y^n o\dot{x}$ + $bnx^m y^{n-1} o\dot{y}$ + $bp x^{m-2} y^n o^2 \dot{x}^2$ + $bq x^m y^{n-2} o^2 \dot{y}^2$ + $bmnx^{m-1} y^{n-1} o^2 \dot{y}\dot{x}$ +, &c. for the Moment of $bx^m y^n$.

5. Therefore the Moment last found being divided by the indefinite Quantity o , will give the Fluxion of $bx^m y^n$ equal to $bmx^{m-1} y^n \dot{x}$ + $bnx^m y^{n-1} \dot{y}$ + $bp x^{m-2} y^n o\dot{x}^2$ + $bq x^m y^{n-2} o\dot{y}^2$ + $bmnx^{m-1} y^{n-1} o\dot{y}\dot{x}$ +, &c. and this is the Fluxion (or Velocity) where-with the foregoing Moment is (or may be) uniformly generated.

6. But since the (Velocity or) Fluxion is required wherewith that Moment first arises, in this Case the Moments $o\dot{x}$ and $o\dot{y}$ will also be just arising and therefore nothing, and consequently o will be nothing, and therefore all the Terms wherein it is found will be nothing:

7. Therefore the Fluxion of $bx^m y^n$ at that Moment of Time is accurately $bmx^{m-1} y^n \dot{x}$ + $bnx^m y^{n-1} \dot{y}$.
Q. E. D.

Otherwise thus.

1. Let $\dot{x}$ and $\dot{y}$ be very small Increments uniformly generated by $\dot{x}$ and $\dot{y}$, in a very small Time.

2. The Quantities $x, y, bx^m y^n$ then at the End of that Time are become $x + \dot{x}, y + \dot{y}$, and $\overbrace{b \times x + \dot{x}}^m \times \overbrace{y + \dot{y}}^n$; and the Increments generated in that Time

Time are $\dot{x}$, $\dot{y}$ and $bmx^{m-1}y^n\dot{x} + bnx^my^{n-1}\dot{y} + bpx^{m-2}y^{n+2}\dot{x}^2 + bqxy^{n-2}\dot{y}^2 + bmnx^{m-1}y^{n-1}\dot{x}\dot{y} + \text{Ec.}$
 $= M$ by Substitution, where note p, q are given Quantities.

3. Now since $\dot{x} : \dot{y} :: \dot{x} : \dot{y} = \frac{\dot{y}\dot{x}}{\dot{x}}$, because the Effects of like Causes are proportional; therefore expunge $\dot{y}$ out of the Value of M , and we shall have the Increment of x to the Increment of $b\dot{x}^m y^n$ as $\dot{x}$ to M or as 1 to $\frac{M}{\dot{x}}$; that is, as $\dot{x}$ to $bmx^{m-1}y^n\dot{x} + bnx^my^{n-1}\dot{y} + bpx^{m-2}y^{n+2}\dot{x}^2 + bmnx^{m-1}y^{n-1}\dot{x}\dot{y} + bqxy^{n-2}\dot{y}^2 + \text{Ec.} = P + Q\dot{x}$, Ec. by Substitution, (putting P for the two first Terms, and $Q\dot{x}$, Ec. for the rest).

4. But $\dot{x}$ is the Fluxion of x , therefore $P + Q\dot{x}$ is the Fluxion of $b\dot{x}^m y^n$. But since this last Fluxion $P + Q\dot{x}$ is not that with which the Increment begins to be generated, which we seek: It is evident that by diminishing the Time, or which is the same thing, diminishing $\dot{x}$, the Fluxion $P + Q\dot{x}$ continually converges to that Fluxion (or Velocity) wherewith the Increment first arises, and before $\dot{x}$ be diminish'd to nothing, is nearer to it than by any given Difference; and therefore by the Axiom when $\dot{x}$ is nothing, and consequently $Q\dot{x}$ nothing, then P or $bmx^{m-1}y^n\dot{x} + bnx^my^{n-1}\dot{y}$ will be the Fluxion of $b\dot{x}^m y^n$. In like Manner it may be demonstrated for any other Quantity. Q. E. D.
 COR. I.

The DOCTRINE

COR. 1. A Fluent can have but one Fluxion : Thus the Fluxion of x can only be $\dot{x}$; of ax^2 , $2ax\dot{x}$, &c.

COR. 2. When the Fluxion of any Quantity is the same with a proposed Fluxion, then that Quantity is its Fluent.

SCHOLIUM.

As clear and evident as this Proposition is, yet it has been censur'd as false and erroneous ; though the Persons that object against its Truth were never able to tell us what the Error is, nor whether the Fluxion of any Quantity is greater or lesser than is assign'd by this Proposition.

For a further Illustration, let us suppose that x and y are uniformly diminished by the very small Incre-

ments $\dot{x}$ and $\dot{y}$. Then by reasoning as before, we shall get the Fluxion (or Velocity) of the contempora-

ry Moment of $bx^m y^n = bmx^{m-1} y^n \dot{x} + bnx^m y^{n-1} \dot{y} -$

$$bpx^{m-2} y^n \dot{x} \dot{x} - bmnx^{m-1} y^{n-1} \dot{x} \dot{y} - \frac{bqx^m y^{n-2} \dot{x} \dot{y}^2}{\dot{x}}$$

&c. = $P - Q\dot{x}$, &c. Now by continually diminish-

ing $\dot{x}$, it is manifest that $P - Q\dot{x}$, &c. will differ from the Fluxion the Moment first arises with, by a Quantity less than any assignable, before $\dot{x}$ be reduced to nothing.

Now when we see that whilst $x + \dot{x}$ converges to x ,

that at the same Time $P + Q\dot{x}$, &c. converges to that Fluxion the Moment arises with, and differs in *Excess* from it by less than any given Difference, and still dif-

fers the less, the less $\dot{x}$ is taken ; and when $\dot{x}$ is infinitely small, that Difference is infinitely small in *Ex-*

cess: And seeing also that whilst $x - \dot{x}$ converges to-
wards

wards x , likewise $P - Q\dot{x}$, &c. approaches to the Fluxion the Moment vanishes with (or the succeeding Moment begins with, being the same the other converg'd to) and differs in *Defect* from it, by less than any given Deference, and still differs the less as $\dot{x}$ is less, and when $\dot{x}$ is infinitely diminish'd, that Difference is infinitely small in *Defect*. If any one after all this should contend, that when $\dot{x}$ is quite vanish'd and become nothing, that the Fluxion of $bx^m y^n$ is not accurately P or $bm x^{m-1} y^n \dot{x} + bnx^n y^{n-1} \dot{y}$; I think no man ought to give himself any Concern at all for such an Adversary, or take any further Pains for his Conviction.

P R O P. II.

If two Fluents or variable Quantities be equal to each other, or in a given Ratio; their Fluxions will be equal, or in the same given Ratio.

And if two fluxionary Quantities are equal or in a given Ratio; their contemporary Fluents will be equal or in the same given Ratio.

For since the Quantities always continue equal, or in a given Ratio; the continually arising Increments, and therefore the Fluxions proportional thereto, will necessarily be equal or in the same given Ratio. And *vice versâ*, the Fluxions or generating Causes being always equal or in a given Ratio; their Effects or contemporary Fluents will therefore be equal, or in the same given Ratio. Q. E. D.

C

COR.

The DOCTRINE

COR. 1. Two fluxionary Quantities may be equal, and their simple Fluents unequal. For (by this Prop.) only the contemporary Fluents can be equal. And therefore

COR. 2. A Fluxion may have an infinite Number of Fluents: thus the Fluent of $\dot{x}$ is x , $a + x$, $b + x$, $c + x$, $x - a$, $x - b$, $x - c$, &c.

COR. 3. If any Fluxion be equal (or nearly equal) to some other Fluxion, in some particular Case; then the one may be substituted for the other in any fluxionary Equation, and their Fluents will be equal in that particular Case, (or nearly so.)

COR. 4. If there be any Relation whatsoever between the Moments of several Quantities in their nascent or evanescent State; the same Relation there is also between the Fluxions of these Quantities.

COR. 5. And therefore in any Equation between the Moments consider'd as arising, the Fluxions may be substituted in their Room; and the contrary.

P R O P. III.

Given an Equation containing the Relation of the flowing Quantities; to determine the Relation of their Fluxions.

SOLUTION.

1. If any Term contains only one variable Quantity; multiply its Index, the Fluxion of the Root, its Power whose Index is lessened by 1, and the Coefficient of the Term continually, for the Fluxion of that Term:

2. If any Term involves several flowing Quantities, observe what Power of any variable Quantity is contain'd

Sect. I. of FLUXIONS.

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tain'd in that Term, and multiply the Fluxion of that Power (found by Art. 1.) by all the other Quantities in the Term; do the same for all the variable Quantities; the Sum of all these Products is the Fluxion of that Term.

3. Repeat the same Operation for all the Terms in the Equation; and the Sum of all gives an Equation containing the Fluxions required.

4. In exponential Equations, or those whose Exponents are variable; let $X, Y, Z, \&c.$ be the hyperbolic Logarithms of $x, y, z, \&c.$ then multiply the Index of any Quantity into the Logarithm of that quantity, and you'll have a logarithmic Equation, whose Fluxion is to be found by the foregoing Rules: Then expunge the logarithmic Fluxions $\dot{X}, \dot{Y},$ or $\dot{Z},$ by substituting their equals $\frac{\dot{x}}{x}, \frac{\dot{y}}{y},$ or $\frac{\dot{z}}{z}$; the Reason of which will appear hereafter by Prob. II. Sect. II.

5. Sometimes it may be convenient to divide the Equation given, by some of the indetermin'd Quantities contained in most of the Terms; or to substitute single Letters for compound Quantities.

EXAMPLE I.

Let $y = a + x - z - b - dx$; then the Fluxion is $\dot{y} = \dot{x} - \dot{z} - d\dot{x}.$

Ex. 2.

Let $x = ay^3$, then the Fluxion of this Equation is $\dot{x} = 3ay^2\dot{y}.$

Ex. 3.

Let $y = ax^2$, the Fluxion is $\dot{y} = 2ax\dot{x}.$ Or in particular Numbers, let $a = 10, \dot{x} = 1$, then $\dot{y} = 20x.$ Now if $x = 1$, then $\dot{y} = 20.$ If $x = 2, 3, 4, \&c.$ then $\dot{y} = 40, 60, 80, \&c.$

Ex. 4.

Let $z = 120bx^n$, the Fluxion is $\dot{z} = 120nbx^{n-1}\dot{x}.$

C 2

Ex.

Ex. 5.

Let $\sqrt{aa - xx} = y$, that is $aa - xx^{\frac{1}{2}} = y$; its Fluxion is $\frac{1}{2} \times -2x\dot{x} \times aa - xx^{\frac{1}{2}}^{-\frac{1}{2}} = \dot{y}$, or $\frac{-x\dot{x}}{\sqrt{aa - xx}} = \dot{y}$. In Numbers thus, let $a=10$, $\dot{x}=1$, then $\dot{y} = \frac{-x}{\sqrt{100 - xx}}$, and if $x=0$, then $\dot{y} = \frac{-0}{10} = 0$. If $x=3$, $\dot{y} = \frac{-3}{\sqrt{91}}$. If $x=6$, then $\dot{y} = \frac{-3}{4}$. If $x=10$, $\dot{y} = \frac{-10}{0} = -\text{Infinity}$; and so of others.

Ex. 6.

Let $v = bx^3y^2$, the Fluxion is $\dot{v} = 3by^2x^2\dot{x} + 2bx^3y\dot{y}$.

Ex. 7.

Let $\frac{x}{y} = v$, that is $x^1y^{-1} = v$; the Fluxion is $y^{-1}\dot{x} - 1 \times x^1y^{-2}\dot{y} = \dot{v}$, or $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2} = \dot{v}$, that is $\frac{y\dot{x} - x\dot{y}}{yy} = \dot{v}$.

Ex. 8.

Let $x^3 - ax^2 + axy - y^3 = 0$. Its Fluxion will be $3x^2\dot{x} - 2ax\dot{x} + ax\dot{y} + ay\dot{x} - 3y^2\dot{y} = 0$.

Ex. 9.

Let this Equation be given $y^2 - a^2 = x\sqrt{aa - xx}$; its Fluxion is $2y\dot{y} = \dot{x}\sqrt{aa - xx} - \frac{x^2\dot{x}}{\sqrt{aa - xx}}$, or $2y\dot{y} = \frac{aa\dot{x} - 2x^2\dot{x}}{\sqrt{aa - xx}}$.

Ex.

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Ex. 10.

Let $2y^3 + x^2y - 2cyz = z^3 - 3yz^2$ be given; then its Fluxion is $6y^2\dot{y} + 2yx\dot{x} + x^2\dot{y} - 2cy\dot{z} - 2cz\dot{y} = 3z^2\dot{z} - 3z^2\dot{y} - 6yz\dot{z}$.

Or thus, divide the given Equation by y , and you'll have $2y^2 + x^2 - 2cz = \frac{z^3}{y} - 3z^2$, whose Fluxion is $4y\dot{y} + 2x\dot{x} - 2c\dot{z} = \frac{3z^2\dot{z}}{y} - \frac{z^3\dot{y}}{y^2} - 6z\dot{z}$; or $4y^3\dot{y} + 2yyx\dot{x} - 2cy^2\dot{z} = 3yz^2\dot{z} - z^3\dot{y} - 6y^2z\dot{z}$.

Ex. 11.

$x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay+xx} = 0$, the Fluxion is $3x^2\dot{x} - 2ay\dot{y} + \frac{3by^2\dot{y}}{a+y} - \frac{by^3\dot{y}}{(a+y)^2} - 2x\dot{x}\sqrt{ay+xx} - \frac{ax^2\dot{y} - 2x^3\dot{x}}{2\sqrt{ay+xx}} = 0$. Or $3x^2\dot{x} - 2ay\dot{y} + \frac{3aby^2\dot{y} + 2by^3\dot{y}}{(a+y)^2} - \frac{4ayx\dot{x} + 6x^3\dot{x} + ax^2\dot{y}}{2\sqrt{ay+xx}} = 0$.

Ex. 12.

Let $\sqrt{ax} + \sqrt{aa-xx} = v$, then will $ax + \sqrt{aa-xx} = vv$, whose Fluxion is $a\dot{x} - \frac{xx\dot{x}}{\sqrt{aa-xx}} = 2v\dot{v}$, whence $\dot{v} = \frac{a\dot{x}}{2v} - \frac{xx\dot{x}}{2v\sqrt{aa-xx}}$, or $\dot{v} = \frac{a\dot{x} - \frac{xx\dot{x}}{\sqrt{aa-xx}}}{2\sqrt{ax} + \sqrt{aa-xx}}$.

Ex. 13.

Let $zy - ax = 0$; its Fluxion will be $\dot{z}y + zy\dot{z} - a\dot{x} = 0$. Or thus, let $s = y$, $t = x$, then $\dot{s} = \dot{y}$, and $\dot{t} = \dot{x}$; and the Equation $zy - ax = zs - at = 0$. whose

whose Fluxion $z\dot{s} + s\dot{z} - a\dot{t} = 0$, that is (restoring the Values of $s, \dot{s}, t$) $z\ddot{y} + \dot{y}\dot{z} - a\ddot{x} = 0$, the same as before.

Ex. 14.

Suppose $2xx\dot{z} = ay\ddot{y}$, and let $\dot{x}$ be a given or constant Quantity, then its Fluxion is $2\dot{x}^2\dot{z} + 2xx\ddot{z} = ay\ddot{y} + a\dot{y}\ddot{y}$.

Or thus, let $\dot{x} = b$, $\dot{z} = t$, $\dot{y} = v$; then the given Equation becomes $2bxt = avv$, whose Fluxion is $2bt\dot{x} + 2bx\dot{t} = av\dot{v} + av\ddot{v}$. And restoring $\dot{z}, \ddot{z}, \dot{x}, \ddot{x}, \dot{y}, \ddot{y}$ for $t, \dot{t}, b, v, \dot{v}$; you'll have $2\dot{z}\dot{x}^2 + 2xx\ddot{z} = a\dot{y}\ddot{y} + ay\ddot{y}$, the Fluxion required.

Ex. 15.

Let $\frac{a\dot{z}\ddot{x}}{\dot{x}^2} + \frac{y\ddot{y}}{a} - \dot{y} = 0$, and suppose $\dot{y}$ constant; then its Fluxion is $\frac{a\dot{z}\ddot{x}}{\dot{x}^2} + \frac{a\ddot{z}\ddot{x}}{\dot{x}^2} - \frac{2a\dot{z}\ddot{x}^2}{\dot{x}^3} + \frac{\dot{y}^2}{a} = 0$.

Ex. 16.

Suppose $2a^3y\ddot{y} - x^2\dot{z}^2 - x^2\dot{z}\ddot{x} = a^3\dot{y}^2 - \frac{a^3\dot{y}^2\ddot{y}}{\dot{z}^2}$, and $\dot{z}$ invariable, its Fluxion will be $2a^3\dot{y}\ddot{y} + 2a^3y\ddot{y} - 2xx\dot{z}^2 - 2xx\dot{z}\ddot{z} - x^2\dot{z}\ddot{x} = 2a^2\dot{y}\ddot{y} - \frac{2a^3\dot{y}\ddot{y}^2}{\dot{z}^2} - \frac{a^3\dot{y}^2\ddot{y}}{\dot{z}^2}$.

Ex. 17.

Let $Y^3 = x$, where Y is the hyperbolic Log. of y ; then its Fluxion is $3Y^2\dot{Y} = \dot{x}$; but $\dot{Y} = \frac{\dot{y}}{y}$; therefore $\dot{x} = \frac{3Y^2\dot{y}}{y}$.

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Ex. 18.

Suppose $y^x = z$, then will $xY = Z$ (where Z, Y are the Hyp. Log. of z, y); the Fluxion of this Equation is $x\dot{Y} + Y\dot{x} = \dot{Z}$; but $\dot{Y} = \frac{\dot{y}}{y}$, and $\dot{Z} = \frac{\dot{z}}{z}$, whence $\frac{x\dot{y}}{y} + Y\dot{x} = \frac{\dot{z}}{z}$; therefore $\dot{z} = \left(\frac{xz\dot{y}}{y} + Yz\dot{x} \right) = xy^{x-1}\dot{y} + Yy^x\dot{x}$.

Ex. 19.

Let $y^{x^v} = z$; then $x^vY = Z$, in Fluxions $x^v\dot{Y} + Y \times \text{Fluxion of } x^v = \dot{Z}$. But (by Ex. 18.) Fluxion of $x^v = vx^{v-1}\dot{x} + Xx^v\dot{v}$; and $\dot{Y} = \frac{\dot{y}}{y}$, also $\dot{Z} = \frac{\dot{z}}{z}$; therefore $\frac{x^v\dot{y}}{y} + Yvx^{v-1}\dot{x} + YXx^v\dot{v} = \frac{\dot{z}}{z} = \frac{\dot{z}}{y^{x^v}}$: Whence $\dot{z} = x^vy^{x^v-1}\dot{y} + vYy^{x^v}x^{v-1}\dot{x} + x^vYXy^{x^v}\dot{v}$.

PROP. IV.

Let the Fluent of $\overline{e + fz^n}^m z^p \dot{z} = A$.

Fluent of $\overline{e + fz^n}^m z^{p+n} \dot{z} = B$.

Fluent of $\overline{e + fz^n}^{m+1} z^p \dot{z} = P$.

Then I say,

$$I. \quad \overline{p+1} \cdot P + \overline{m+1} \times n f B = z^{p+1} \overline{e + fz^n}^{m+1}.$$

$$II. \quad P = eA + fB.$$

DE-

DEMONSTRATION.

For I. $\overline{p+1} \cdot \dot{P} + \overline{m+1} \cdot n\dot{f}B = \overline{p+1} \cdot z^p \dot{z} \cdot \overline{e+fz^n}^{m+1}$
 $+ \overline{m+1} \cdot n\dot{f}z^{p+n} \dot{z} \cdot \overline{e+fz^n}^m$

whose Fluent, by Cor. 2. Prop. I. is

$$\overline{p+1} \times P + \overline{m+1} \times n\dot{f}B = z^{p+1} \overline{e+fz^n}^{m+1}.$$

II. $\overline{e+fz^n}^{m+1} z^p \dot{z} = \overline{e+fz^n} \times \overline{e+fz^n}^m z^p \dot{z} =$
 $e z^p \dot{z} \times \overline{e+fz^n}^m + \overline{e+fz^n}^m z^{p+n} \dot{z} \times e + \overline{e+fz^n}^m z^{p+n} \dot{z} \times f z^n.$ That is

$\dot{P} = e\dot{A} + f\dot{B}$, and the Fluent is $P = eA + fB$.

Q. E. D.

COR. 1. Hence if any one of the Fluents A, B, P , be given, the other two will be found. Therefore,

$$\text{COR. 2. } B = \frac{z^{p+1} \cdot \overline{e+fz^n}^{m+1} - \overline{p+1} \times eA}{p+1+mn+n \times f}$$

$$\text{COR. 3. } A = \frac{z^{p+1} \overline{e+fz^n}^{m+1} - \overline{p+1+mn+n} \cdot fB}{p+1 \times e}$$

$$\text{COR. 4. } P = \frac{\overline{m+1} \times neA + z^{p+1} \overline{e+fz^n}^{m+1}}{p+1+mn+n}$$

$$\text{COR. 5. } A = \frac{\overline{p+1+mn+n} \cdot P - z^{p+1} \overline{e+fz^n}^{m+1}}{m+1 \times ne}$$

PROP. V.

Let $e + fz^n + gz^{2n} + bz^{3n} \&c. = V$. $t =$ Number of Terms in V .

And the Fluent of $V^m z^p \dot{z} = A$.

Fluent of $V^m z^{p+n} \dot{z} = B$.

Fluent of $V^m z^{p+2n} \dot{z} = C$.

Fluent of $V^m z^{p+3n} \dot{z} = D$, $\&c.$ continued

to t Quantities.

And Fluent of $V^{m+1} z^p \dot{z} = P$. Then will,

$$I. \overline{p+1} \cdot eA + \overline{p+1+mn+n} \cdot fB + \left. \begin{array}{l} \overline{p+1+2mn+2n} \cdot gC + \overline{p+1+3mn+3n} \cdot hD \\ +, \&c. \text{ continu'd to } t \text{ Terms.} \end{array} \right\} = z^{p+1} V^{m+1}.$$

$$II. P = eA + fB + gC + hD + \&c. \text{ to } t \text{ Terms.}$$

DEMONSTRATION.

Let $p+1=r$, $m+1=s$. And putting the first Equation into Fluxions, there is $re\dot{A} + \overline{r+sn} \times f\dot{B} + \overline{r+2sn} \times g\dot{C} + \overline{r+3sn} \times h\dot{D} \&c. = rz^p \dot{z} V^{m+1} + z^{p+1} V^m \dot{V}$; that is (restoring the Values of V , $\dot{V}$, $\dot{A}$, $\dot{B}$, $\&c.$ and dividing by $z^p V^m \dot{z}$) $\overline{re+r+sn} \times fz^n + \overline{r+2sn} \cdot gz^{2n} + \overline{r+3sn} \cdot hz^{3n} \&c. = r \cdot e + fz^n + gz^{2n} + hz^{3n} \&c. + sz \times n fz^{n-1} + 2ngz^{2n-1} + 3nhz^{3n-1} \&c.$ where both Sides of the Equation being manifestly equal, 'tis evident the Quantities from whence they were derived, that is, the Fluents in Equation the 1st, must also be equal. After the same Manner is the second Equation very easily demonstrated. Q.E.D.

COR. Hence if $t-1$ of the Fluents P , A , B , C , $\&c.$ be given; the rest will be found.

PROP. VI.

Put $e + fz^n = V$, $k + lz^n = Y$, and

the Fluent of $z^p \dot{z} V^m Y^q = A$.

Fluent of $z^{p+n} \dot{z} V^m Y^q = B$.

Fluent of $z^{p+2n} \dot{z} V^m Y^q = C$.

Also Fluent of $z^p \dot{z} V^{m+1} Y^q = P$.

Fluent of $z^{p+n} \dot{z} V^{m+1} Y^q = Q$.

Then it will be

D

I: $\overline{p+1}$

$$\begin{aligned}
 \text{I. } & \left. \begin{aligned} & \overline{p+1} \times ekA \\ & + \overline{p+1+mn+n} \times fk \\ & + \overline{p+1+qn+n} \times el \\ & + \overline{p+1+mn+qn+2n} \cdot flC \end{aligned} \right\} B \quad \left. \vphantom{\begin{aligned} & \overline{p+1} \times ekA \\ & + \overline{p+1+mn+n} \times fk \\ & + \overline{p+1+qn+n} \times el \\ & + \overline{p+1+mn+qn+2n} \cdot flC \end{aligned}} \right\} = z^{p+1} V^{m+1} Y^{q+1} \\
 \text{II. } & P = eA + fB. \\
 \text{III. } & Q = eB + fC.
 \end{aligned}$$

DEMONSTRATION.

Let $p+1=r$, $m+1=s$, $q+1=t$; and putting the first Equation into Fluxions, we have

$$rek\dot{A} + \left. \begin{aligned} & \overline{r+sn} \times \dot{fk} \\ & + \overline{r+tn} \times \dot{el} \end{aligned} \right\} \dot{B} + \overline{r+sn+tn} \times fl\dot{C} =$$

$$rz^p \dot{z} V^{m+1} Y^{q+1} + sz^{p+1} Y^{q+1} V^m \dot{V} + tz^{p+1} V^{m+1} Y^q \dot{Y}$$

Then restoring the Values of $\dot{A}$, $\dot{B}$, $\dot{C}$, $\dot{V}$, $\dot{Y}$, and dividing by $z^p V^m Y^q$, and then expunging V , Y ,

$$\text{we have } rek + \left. \begin{aligned} & \overline{r+sn} \times \dot{fk} \\ & + \overline{r+tn} \times \dot{el} \end{aligned} \right\} z^n + \overline{r+sn+tn} \times flz^n$$

$$= r \times \overline{e+fz^n} \times \overline{k+lz^n} + snfz^n \times \overline{k+lz^n} + tnlz^n \times \overline{e+fz^n},$$

an Equation whose two Sides (when reduced) being manifestly the same, argues the Equation from whence it was deduced to be a true Equation. And the same may be easily shown of the other two Equations. Q. E. D.

COR. Hence if any 2 of the Fluents A , B , C , P , Q be given; the rest may be found.

P R O P. VII.

$$\text{Let } e + fz^n + gz^{2n} + bz^{3n} \&c. = V.$$

$$k + lz^n + rz^{2n} + sz^{3n} \&c. = Y.$$

t = Number of Terms in V , τ = Number of Terms in Y .

And

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And the Fluent of $z^p \dot{z} V^m Y^q = A.$

Fluent of $z^{p+n} \dot{z} V^m Y^q = B.$

Fluent of $z^{p+2n} \dot{z} V^m Y^q = C.$

Fluent of $z^{p+3n} \dot{z} V^m Y^q = D.$ &c. continu'd
to $t + \tau - 1$ Quantities.

Also Fluent of $z^p \dot{z} V^{m+1} Y^q = P.$

Fluent of $z^{p+n} \dot{z} V^{m+1} Y^q = Q.$

Fluent of $z^{p+2n} \dot{z} V^{m+1} Y^q = R.$ &c. con-
tinu'd to τ Quantities.

Then I say,

$$I. \overline{p+1} \times ekA \left. \begin{array}{l} + \overline{p+1+qn+n} \times el \\ + \overline{p+1+mn+n} \times fk \end{array} \right\} B.$$

$$\left. \begin{array}{l} + \overline{p+1+2qn+2n} \times er \\ + \overline{p+1+mn+qn+2n} \cdot fl \\ + \overline{p+1+2mn+2n} \times gk \end{array} \right\} \left. \begin{array}{l} + \overline{p+1+3qn+3n} \times es \\ C + \overline{p+1+mn+2qn+3n} \cdot fr \\ + \overline{p+1+2mn+qn+3n} \cdot gl \\ + \overline{p+1+3mn+3n} \times bk \end{array} \right\} D$$

+ &c. (continued to $t + \tau - 1$ Terms) =
 $z^{p+1} V^{m+1} Y^{q+1}.$

II. $P = eA + fB + gC + hD$ &c. to t Terms.

III. $Q = eB + fC + gD + hE$ &c.

IV. $R = eC + fD + gE + hF$ &c.

V. $S = eD + fE + gF + hG$ &c.

&c. continued to τ Equations.

This is demonstrated like the foregoing.

COR. Hence if $t + \tau - 2$ of the Fluents, $P, Q, R,$
 $A, B, C,$ &c. are given, the rest will be found.

P R O P. VIII.

Let $V = e + fz^n$. $Y = k + lz^n$. $X = s + tz^n$.

And the Fluent of $z^p \dot{z} V^m Y^q X^r = A$.

Fluent of $z^{p+n} \dot{z} V^m Y^q X^r = B$.

Fluent of $z^{p+2n} \dot{z} V^m Y^q X^r = C$.

Fluent of $z^{p+3n} \dot{z} V^m Y^q X^r = D$.

Also Fluent of $z^p \dot{z} V^{m+1} Y^q X^r = P$.

Fluent of $z^{p+n} \dot{z} V^{m+1} Y^q X^r = Q$.

Fluent of $z^{p+2n} \dot{z} V^{m+1} Y^q X^r = R$.

Then it will be

$$\begin{array}{l}
 \text{I. } \frac{\quad}{p+1} \times e k s A + \frac{p+1+mn+n}{\quad} \times f k s \\
 \quad + \frac{p+1+qn+n}{\quad} \times e l s \quad \left. \vphantom{\frac{p+1+mn+n}{\quad}} \right\} B \\
 \quad + \frac{p+1+rn+n}{\quad} \times e k t \\
 \left. \begin{array}{l}
 + \frac{p+1+mn+qn+2n}{\quad} f l s \\
 + \frac{p+1+mn+rn+2n}{\quad} f k t \\
 + \frac{p+1+qn+rn+2n}{\quad} e l t
 \end{array} \right\} C + \frac{p+1+mn+qn+rn+3n}{\quad} \\
 \times f l t D = z^{p+1} V^{m+1} Y^{q+1} X^{r+1}.
 \end{array}$$

$$\text{II. } P = eA + fB.$$

$$\text{III. } Q = eB + fC.$$

$$\text{IV. } R = eC + fD.$$

The Truth of this is shewn as the rest.

COR. Hence if (any) three of the Fluents A, B, C, D, P, Q, R be given, all the rest will be found.

S C H O L I U M.

In the five last Propositions it must be observed, that any of the Fluents A, B, C; P, Q, R, &c. vanishes out of the Equation when its Coefficient is 0: And therefore in such a Case that Fluent cannot be found by any of these Propositions.

If any one should ask how these Propositions are found out; the Answer is, by the common Analysis. For taking any Equation in these Propositions, or assuming any Equation among the Fluents A, B, C, P, Q, R, at Pleasure, and affecting them with unknown Coefficients; then putting that Equation into Fluxions (substituting the Values of A, B, P, &c.) and then comparing the homologous Terms, these Coefficients will easily be determin'd, if the Thing is possible. And thus you may find out other Propositions of this Kind.

P R O P. IX.

To transform a given Fluxion into another more simple one.

R U L E.

1. For any compound Quantity in the given Fluxion put a new Letter or variable Quantity, by Help of which expunge the other variable Quantity and its Fluxion out of the given fluxionary Quantity; so will you have a new Fluxion instead of the former. If this be not simple enough, assume another variable Quantity instead of any compound Quantity contain'd in this last, and expunge the former assum'd Quantity and its Fluxion. Proceed thus till the transform'd Fluxion be as simple as possible.

2. Sometimes a compound Fluxion may be dissolved into several other simpler ones, thus; assume two or more such fluxionary Quantities, as you conceive will, when duly reduc'd, make up the given Fluxion, and let them be affected with indetermin'd Coefficients; then reducing them to the Form of the given

given Fluxion, these Coefficients will be easily determin'd by comparing the homologous Terms.

3. Or in any fluxionary Equation of x and y , substitute some compound Quantity for one of them, such as zx , or $z^n x$, for y ; or $x\sqrt{zz-1}$ for y , or zx for y , or any such like, that by Substitution will make it simpler. For the whole Design of transforming any Quantity into another is in order to make it more simple, that the Fluent may be the easier found.

EX. 1.

Let $\overline{e + fz^n}^m \times \overline{g + bz^n}^s z^{rn-1} \dot{z}$ be proposed. Assume

sume $v = e + fz^n$, then $z^n = \frac{v-e}{f}$, and $z^{rn} = \frac{v-e}{f^r}$,

and $z^{rn-1} \dot{z} = \frac{\overline{v-e}^{r-1} \dot{v}}{nf^r}$, and $g + bz^n = \frac{fg - be + bv}{f}$

$= \frac{p + bv}{f}$, putting $p = fg - eb$. Therefore the

given Fluxion $\overline{e + fz^n}^m \times \overline{g + bz^n}^s z^{rn-1} \dot{z} = \frac{\overline{v-e}^{r-m}}{nf^{r+1}}$

$\times \overline{p + bv}^s v^m \dot{v}$.

EX. 2.

There is given $\frac{z^{rn-1} \dot{z}}{\overline{k + lz^n \times e + fz^n + gz^{2n}}^{2n}}$; assume

$= k + lz^n$, then $z^n = \frac{v-k}{l}$, and $z^{rn} = \frac{\overline{v-k}^r}{l^r}$,

$z^{rn-1} \dot{z} = \frac{\overline{v-k}^{r-1} \dot{v}}{nl^r}$; put $p = \sqrt{ff - 4eg}$, and then

$\frac{1}{e + fz^n + gz^{2n}} = \frac{g}{p \cdot \frac{f-p}{2} + gz^{2n}} - \frac{g}{p \cdot \frac{f+p}{2} + gz^{2n}} \times \frac{1}{f}$

as will be evident by reducing the two Terms to a common Denominator. Therefore, putting $a =$

$\frac{f-p}{2} l - gk$, and $b = \frac{f+p}{2} l - gk$, the given Fluxion

$$\text{becomes } \frac{g}{p} \times \frac{\overline{v-k}^{r-1} \overline{v-1} \dot{v}}{nl^{r-1} \times \overline{a+gv}} - \frac{\overline{v-k}^{r-1} \overline{gv-1} \dot{v}}{npl^{r-1} \times \overline{b+gv}}.$$

Ex. 3.

Let $\frac{e + fz^n z^{\lambda n-1} \dot{z}}{g + bz^n + r} = \dot{F}$ be given. Put $v = g +$

bz^n . Then $z^n = \frac{v-g}{b}$, and $e + fz^n = \frac{fv-p}{b}$ (put-

ting $p = fg - eb$) and $z^{\lambda n-1} \dot{z} = \frac{\overline{v-g}^{\lambda-1} \dot{v}}{nb^\lambda}$; whence

$$\dot{F} = \frac{\overline{fv-p}^m \times \overline{v-g}^{\lambda-1} \overline{v-m-r} \dot{v}}{nb^{\lambda+m}} = \frac{\overline{1-gv-1}^{\lambda-1}}{nb^m + \lambda} \times$$

$$\overline{f-pv-1}^m \overline{v^{\lambda-r-1}} \dot{v}.$$

Again, let $f - pv^{-1} = y$; then $v^{\lambda-r-1} \dot{v} =$

$$\frac{\overline{f-y}^{r-\lambda-1} \dot{y}}{p^{r-\lambda}}, \text{ and } 1 - gv^{-1} = \frac{gy - eb}{p}; \text{ whence } \dot{F}$$

$$= \frac{\overline{gy-eb}^{\lambda-1}}{nb^m + \lambda p^{r-1}} \times \overline{f-y}^{r-\lambda-1} y^m \dot{y}.$$

Otherwise.

Since $\frac{\overline{1-gv-1}^{\lambda-1}}{nb^m + \lambda} \times \overline{f-pv-1}^m \overline{v^{\lambda-r-1}} \dot{v}$ is (by

first multiplying and then dividing, by $1 - gv^{-1}$) =

$$\frac{\overline{1-gv-1}^\lambda}{nb^m + \lambda} \times \overline{f-pv-1}^m \overline{v^{\lambda-r-1}} \dot{v} + \frac{g \cdot \overline{1-gv-1}^{\lambda-1}}{nb^m + \lambda}$$

$\times \overline{f-pv-1}^m \overline{v^{\lambda-r-2}} \dot{v}$; therefore expunging v and it becomes

becomes $\dot{F} = \frac{gy - eb^\lambda}{nb^{m+\lambda}p^r} \times f - y^{r-\lambda-1} y^m \dot{y} + \frac{g \cdot gy - eb^{\lambda+1}}{nb^{m+\lambda}p^r}$
 $\times f - y^{r-\lambda} y^m \dot{y}$, and $y = f - pv^{-1} = \frac{e + fz^n}{g + bz^n} b$.

Ex. 4.

Suppose $e + fz^n + gz^{2n} z^{r-1} \dot{z} = \dot{F}$, put $e + fz^n + gz^{2n} = v$, $p = \frac{1}{2}ff - eg$, then $z^n = \frac{-\frac{1}{2}f + \sqrt{p + gv}}{g}$,

and $z^{r-1} \dot{z} = \frac{-\frac{1}{2}f + \sqrt{p + gv}}{g^r} \times \frac{g \dot{v}}{2n\sqrt{p + gv}}$;

therefore $\dot{F} = \frac{-\frac{1}{2}f + \sqrt{p + gv}}{2ng^{r-1}} \times \frac{v^m \dot{v}}{\sqrt{p + gv}}$. If

this be not thought simple enough, suppose,

Again $x = \sqrt{p + gv}$, then $v = \frac{xx - p}{g}$, and $v^{m+1} = \frac{xx - p}{g^{m+1}}$, and $v^m \dot{v} = \frac{xx - p}{g^{m+1}} \times 2x \dot{x}$; therefore $\dot{F} = \frac{x - \frac{1}{2}f}{ng^{m+r}} \times x^2 - p^m \dot{x}$. Here $x = \sqrt{p + gv} = \frac{1}{2}f + gz^n$.

Ex. 5.

To transform the Fluxion $\frac{z^{\lambda n - 1} \dot{z}}{e + fz^n + gz^{2n}}$, assume

$$\frac{gz^{\lambda n - 1} \dot{z}}{ge + gfz^n + ggz^{2n}} = \frac{Az^{\lambda n - 1} \dot{z}}{B + gz^n} + \frac{Cz^{\lambda n - 1} \dot{z}}{D + gz^n} =$$

(by reducing to a common Denominator)

$$\frac{+AD}{+BC} \frac{z^{\lambda n - 1} \dot{z}}{z^{\lambda n - 1} \dot{z}} + \frac{Ag}{+Cg} \frac{z^{\lambda n + n - 1} \dot{z}}{z^{\lambda n + n - 1} \dot{z}}$$

$$\frac{BD + Bg}{+Dg} \frac{z^n + ggz^{2n}}{z^n + ggz^{2n}}; \text{ then comparing the}$$

homologous

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homologous Terms, $Ag + Cg = 0$, $AD + BC = g$, $BD = ge$, $Bg + Dg = fg$; whence is had $C = -A$, $D + B = f$, $D - B = \sqrt{ff - 4eg} = p$; and thence $A = \frac{g}{p}$,

$$B = \frac{f-p}{2}, C = \frac{-g}{p}, D = \frac{f+p}{2}. \text{ And therefore,}$$

$$\frac{z^{\lambda n - 1} \dot{z}}{e + fz^n + gz^{2n}} = \frac{g}{p} \times \frac{z^{\lambda n - 1} \dot{z}}{\frac{f-p}{2} + gz^n} - \frac{g}{p} \times \frac{z^{\lambda n - 1} \dot{z}}{\frac{f+p}{2} + gz^n}.$$

Otherwise thus.

Put $z^n = y - \frac{f}{2g}$, then $z^{\lambda n - 1} \dot{z} = \frac{\dot{y}}{n} \times \left[y - \frac{f}{2g} \right]^{\lambda - 1}$. Whence the fluxional Quantity pro-

posed becomes $\frac{\frac{1}{n} \times \left[y - \frac{f}{2g} \right]^{\lambda - 1}}{e - \frac{ff}{4g} + gyy}$, which will be

in rational Terms when λ is an Integer.

Ex. 6.

Let $\frac{z^{\lambda n - 1} \dot{z}}{e + fz^n + gz^{2n}}$ be given, where λ is half of an odd Number, and $4eg$ greater than ff .

Assume $\frac{gz^{\lambda n - 1} \dot{z}}{eg + fgz^n + ggz^{2n}} = \frac{Az^{\lambda n - \frac{1}{2}n - 1} \dot{z}}{C - Dz^{\frac{1}{2}n} + gz^n} -$

$$\frac{Bz^{\lambda n - \frac{1}{2}n - 1} \dot{z}}{C + Dz^{\frac{1}{2}n} + gz^n} = (\text{by Reduction})$$

$$\frac{+ ACz^{\lambda n - \frac{1}{2}n - 1} \dot{z} + ADz^{\lambda n - 1} \dot{z} + Agz^{\lambda n + \frac{1}{2}n - 1} \dot{z} - BC + BD - Bg}{CC + 2Cg z^n + ggz^{2n}}.$$

Now comparing the homologous Terms, we have
 $AC - BC = 0 = Ag - Bg$, $AD + BD = g$, $CC = eg$,
E $2Cg$

$2Cg - DD = fg$. From whence we have $C = \sqrt{eg}$,

$D = \sqrt{2g\sqrt{eg} - fg}$, $A = \frac{g}{2D} = B$. Whence

$$\frac{z^{\lambda n - 1} \dot{z}}{e + fz^n + gz^{2n}} = \frac{g}{2D} \times \frac{z^{\lambda n - \frac{1}{2}n - 1} \dot{z}}{\sqrt{eg} - Dz^{\frac{1}{2}n} + gz^n} - \frac{g}{2D} \times \frac{z^{\lambda n - \frac{1}{2}n - 1} \dot{z}}{\sqrt{eg} + Dz^{\frac{1}{2}n} + gz^n}$$
: both which Terms may be transform'd into others still more simple, after the Manner of the fourth Example.

Ex. 7.

Let $\frac{z^{\lambda n - 1} \dot{z}}{k + lz^n \times e + fz^n + gz^{2n}}$ be proposed, where λ is half an odd Number, and $4eg$ greater than ff .

Assume for it $\frac{Az^{\lambda n - n - 1} \dot{z} + Bz^{\lambda n - 1} \dot{z}}{e + fz^n + gz^{2n}} + \frac{Dz^{\lambda n - n - 1} \dot{z}}{k + lz^n}$,

or for Brevity Sake (dividing by $z^{\lambda n - n - 1} \dot{z}$)

$$\frac{z^n}{k + lz^n \times e + fz^n + gz^{2n}} = \frac{A + Bz^n}{e + fz^n + gz^{2n}} + \frac{D}{k + lz^n} = \frac{A + Bk + Bk + Bz^{2n}}{e + fz^n + gz^{2n}} + \frac{D}{k + lz^n} = \frac{A + Bk + Bz^{2n} + Dk + Df + Dg}{e + fz^n + gz^{2n}} + \frac{D}{k + lz^n}$$

(by Reduction) $\frac{A + Bk + Bz^{2n} + Dk + Df + Dg}{k + lz^n \times e + fz^n + gz^{2n}}$; and comparing the homologous Terms, $Ak + De = 0$, $Al + Bk + Df = 1$, $Bl + Dg = 0$, whence (putting $t = ell - fkl + gkk$) we have $D = \frac{-lk}{t}$, $A = \frac{el}{t}$, $B = \frac{gk}{t}$. Whence

$$\frac{z^{\lambda n - 1} \dot{z}}{k + lz^{2n} \cdot e + fz^n + gz^{2n}} = \frac{elz^{\lambda n - n - 1} \dot{z} + gkz^{\lambda n - 1} \dot{z}}{t \times e + fz^n + gz^{2n}} - \frac{lkz^{\lambda n - n - 1} \dot{z}}{t \times k + lz^n}$$

; and the first Part may be transformed again into others more simple by the sixth Example.

I shall here add some more Examples, to illustrate the Method of proceeding.

Ex.

Ex. 8.

To transform the Fluxion $\frac{z^{\delta n-1} \dot{z}}{p+qz^n+z^{2n} \times r+sz^n+z^{2n}}$

assume $\frac{1}{p+qz^n+z^{2n} \times r+sz^n+z^{2n}} = \frac{A+Bz^n}{p+qz^n+z^{2n}} + \frac{C+Dz^n}{r+sz^n+z^{2n}}$. Then multiplying cross ways as before, we get

$$\left. \begin{array}{l} Ar + Asz^n + Az^{2n} \\ + Br + Bs + Bz^{3n} \\ + Cp + Cq + C \\ + Dp + Dq + D \end{array} \right\} = 1.$$

Hence we shall get $Ar + Cp = 1$. $As + Br + Cq + Dp = 0$, $A + Bs + C + Dq = 0$, and $B + D = 0$. Hence may be had the Values of A, B, C, D ; and then

$$\text{the Fluxion becomes } \frac{Az^{\delta n-1} \dot{z} + Bz^{\delta n+n-1} \dot{z}}{p+qz^n+z^{2n}} + \frac{Cz^{\delta n-1} \dot{z} + Dz^{\delta n+n-1} \dot{z}}{r+sz^n+z^{2n}}.$$

Ex. 9.

To transform the Fluxion $\frac{\dot{y}}{a^3+y^3}$. Put it =

$\frac{A+By}{aa-ay+yy} \dot{y} + \frac{cy}{a+y}$; then multiplying alternately, and equating the homologous Terms; there is found $A = \frac{2}{3a}$, $B = \frac{-1}{3aa}$, $C = \frac{1}{3aa}$. Whence

$$\frac{\dot{y}}{a^3+y^3} = \frac{1}{3aa} \times \frac{2a-y}{aa-ay+yy} \dot{y} + \frac{1}{3aa} \times \frac{\dot{y}}{a+y}.$$

The first Term may be further transformed thus;

$$\text{put } y = z + \frac{1}{2}a, \text{ and there comes out } \frac{1}{3aa} \times \frac{\frac{3}{2}a-z}{\frac{3}{4}aa+zz} \dot{z}; \text{ therefore } \frac{\dot{y}}{a^3+y^3} = \frac{1}{3aa} \times \frac{\frac{3}{2}a-z}{\frac{3}{4}aa+zz} \dot{z} + \frac{1}{3aa} \times \frac{\dot{y}}{a+y}.$$

E 2

Ex.

Ex. 10.

To transform the Fluxion $\frac{dz^m \dot{z}}{1 + az^{2n+1}}$; m, n being Integers. Suppose $az^{2n+1} = x^{2n+1}$; and the Fluxion becomes $\frac{d}{a^{2n+1}} \times \frac{x^m \dot{x}}{1 + x^{2n+1}}$. To transform

form $\frac{x^m \dot{x}}{1 + x^{2n+1}}$, put $1 + x^{2n+1} = 1 + x \times 1 - x + xx - x^3 + x^4 - \dots + x^{2n}$. and $1 - x + x^2 - x^3 \dots + x^{2n} = 1 + px + x^2 \times 1 + qx + xx \times 1 + rx + xx$, &c. The latter Side of the Equation being multiply'd, and compared with the homologous Terms of the first Side, will give the Coefficients p, q, r , &c. Then

Let $\frac{x^m \dot{x}}{1 + x^{2n+1}} = \frac{x^m \dot{x}}{1 + x \times 1 + px + xx \times 1 + qx + xx \times 1 + rx + xx}$
 $\mathcal{E}c. = \dot{x} \times \text{into } \frac{a}{1+x} + \frac{b+cx}{1+px+xx} + \frac{d+ex}{1+qx+xx}$
 $+ \frac{f+gx}{1+rx+xx} \mathcal{E}c.$ This reduced to a common Denominator, and the homologous Terms of the Numerators compared, will determine the Quantities a, b, c, d , &c.

And the same Way the Fluxion of $\frac{x^m \dot{x}}{1 - x^{2n}}$ is transformed, by putting $1 - x^{2n} = 1 + x \times 1 - x + x^2 - x^3 + \dots - x^{2n-1} = 1 + x \times 1 + px + xx \times 1 + qx + xx \times 1 + rx + xx$, $\mathcal{E}c.$ and proceeding in all Respects as before: where m, n are Integers.

Ex. 11.

To transform the Fluxion $\frac{x^m \dot{x}}{1 - x^n}$, m, n being Integers. Put $1 - x^n = 1 - x \times 1 + x + xx + x^3 \dots + x^{n-1}$,

x^{n-1} , and $1+x+x^2+x^3 \dots + x^{n-1} = 1+px+xx \times 1+qx+xx \times 1+rx+xx$, &c. the latter Side of the Equation being multiply'd, and the homologous Terms of both Sides compared, there will be found the Coefficients p, q, r , &c. Then

Let $\frac{x^m \dot{x}}{1-x^n} = \frac{x^m \dot{x}}{1-x \times 1+px+xx \times 1+qx+xx, \&c.}$

$= \dot{x} \times \text{into } \frac{a}{1-x} + \frac{b+cx}{1+px+xx} + \frac{d+ex}{1+qx+xx}$ &c. This being reduced to a common Denominator, and the homologous Terms of the Numerator compared, will determine the Coefficients a, b, c, d , &c.

Ex. 12.

To transform $\frac{x^m \dot{x}}{1-x^{2n}}$, m, n being Integers.

Put $1-x^{2n} = 1-xx \times 1+xx+x^4+\dots+x^{m-2}$, and $1+xx+x^4+\dots+x^{2n-2} = 1+px+xx \times 1+qx+xx$ &c. which multiply'd, and the homologous Terms compared, give p, q, r , &c. Then let $\frac{x^m \dot{x}}{1-x^{2n}}$

$= \frac{x^m \dot{x}}{1-xx \times 1+px+xx \times 1+qx+xx} = \dot{x} \times$

$\frac{a}{1-xx} + \frac{b+cx}{1+px+xx} + \frac{d+ex}{1+qx+xx}$ &c. Which

reduced to a common Denominator, and the homologous Terms of the Numerator compared, give a, b, c , &c.

And the Fluxion $\frac{x^m \dot{x}}{1+x^{4n+2}}$ is transform'd by

putting $1+x^{4n+2} = 1+xx \times 1-x^2+x^4-x^6 \dots + x^{4n} = 1+xx \times 1+px+xx \times 1+qx+xx$, &c. and proceeding in all Respects as before.

Likewise

Likewise $\frac{x^m \dot{x}}{1 - x^{4n+2}}$ is transformed, by assuming the same Series as the last; that is, $1 - x^{4n+2} = \frac{1+xx}{1+qx+xx} \times \frac{1-x^2+x^4 \dots - x^{4n-2}}{1+px+xx}$ &c. and proceeding the same Way.

Ex. 13.

To transform $\frac{z^{\frac{1}{3}n-1} \dot{z}}{e \pm fz^n + gz^{2n}}$, when $n = 3$
 $4eg > ff$; put $\frac{e}{g} = R^6$, $\frac{f}{2gR^3} = b$, then we have
 $\times \frac{\dot{z}}{R^6 \pm 2bR^3z^3 + z^6}$. Now if $A = \text{Arch}$ whose
 natural Cosine is b , $\frac{A}{3} = B$, $\frac{360}{3} \text{ or } 120 = C$; and
 s, t, v be the natural Cosines of $B, B+C, B+2C$
 all taken from the Tables; then Rb, Rs, Rt, Rv
 will be the Cosines to the radius R ; then will R^6
 $2bR^2z^3 + z^6 = \frac{RR - 2Rs z + z z \times RR - 2t Rz + z z \times RR - 2v Rz + z z}{RR - 2Rv z + z z}$. Then the Fluxion is transform'd

$$\frac{1}{g} \times \dot{z}$$

$$\frac{R^2 - 2Rs z + z z \times R^2 - 2t Rz + z z \times R^2 - 2v Rz + z z}{R^2 - 2Rv z + z z}$$

Again, let this be put $= \frac{K + kz}{R^2 - 2Rs z + z z} + \frac{L + lz}{R^2 - 2Rt z + z z}$

$$+ \frac{M + mz}{R^2 - 2Rv z + z z} : \times \frac{\dot{z}}{g}; \text{ and the Quantities } K, L, l, \text{ \&c. will be found by bringing all the Terms to a common Denominator and equating the Coefficients of the homologous Terms. Thus if } R \text{ be made } = 1, \text{ and } q = -2Rs, r = -2Rt, f = -2Rv$$

the

from there will come out $K = \frac{qq - I}{q - r \times q - f}$, $k =$

$$\frac{q}{q-r \times q-f}; L = \frac{rr-1}{r-q \times r-f}, l = \frac{r}{r-q \times r-f};$$

$$M = \frac{\int f - I}{\overline{f - q} \times \overline{f - r}}, m = \frac{f}{\overline{f - q} \times \overline{f - r}}. \text{ In which}$$

R may be restored, and the Terms made homogeneous; and for that Purpose dividing the Values of K, L, M, by R^4 , and k, l, m , by R^3 .

Ex. 14.

To transform the Fluxional Quantity $\frac{z^{\lambda-1} \cdot z}{e \pm fz^n + z^{2n}}$;

where δ , λ , η , are Integers, and $\delta\delta$ less than $4e$.
Put $x^\eta = x^\lambda$, and then the Fluxion becomes $\frac{\lambda}{\eta} \times$

$$\frac{x^{\delta-1} \dot{x}}{\pm f x^{\lambda} + x^{2\lambda}}.$$

Again, to transform $\frac{x^{\delta-1} \dot{x}}{e \pm f x^{\lambda} + x^{2\lambda}}$; let $e^{\frac{1}{2}\lambda} = R$,

$$\frac{f}{2R^{\lambda}} = b, \quad A = \text{Arch whose nat. Cofign is } \mp b, \quad \frac{A}{3}$$

$$= B, \frac{360}{\lambda} = C; \text{ and } S, T, V, W, \&c. \text{ the nat.}$$

Cofines of B , $B+C$, $B+2C$, $B+3C$, &c. to λ
 Terms; all taken from the Tables: Then Rb , RS ,
 RT , &c. will be the Cofines to the Radius R , and then

$$K, \frac{\pm fx^\lambda + x^{2\lambda}}{\times RR - zRTx + xx} \text{ or } \frac{R^{2\lambda} \pm 2bR^\lambda x^\lambda + x^{2\lambda}}{\times RR - zRVx + xx} = \overline{RR - zRSx + xx} \text{ } \mathcal{E}c. \text{ to } \lambda \text{ Terms.}$$

Therefore the Fluxion now becomes

$$RR - 2RSx + xx \times RR - 2RTx + xx \times \mathcal{C}.$$

Lastly,

Lastly, put this last = $\frac{K + kx}{RR - 2RSx + xx}$

$\frac{L + lx}{RR - 2RTx + xx}$ &c. $\times$ into $x^{\lambda-1}\dot{x}$, or rather $\times$ into $\frac{x^{\lambda-1}\dot{x}}{R^{2\lambda-2}}$; to make the Quantities homogeneous. Q

for Brevity's Sake, let $R = 1$, $q = -2Rs$, $r = -2RT$, $s = -2RV$, $t = -2RW$, &c. The

$x^{\lambda-1}\dot{x} \times \frac{1}{1+qx+xx \times 1+rx+xx \times 1+sx+xx}$, &c.

$x^{\lambda-1}\dot{x} \times: \frac{K + kx}{1 + qx + xx} + \frac{L + lx}{1 + rx + xx} + \frac{M + mx}{1 + sx + xx}$

&c. Then, by bringing the Terms to a common Denominator, K , k , L , &c. will be determined by comparing the Coefficients as usual. And at last R may be restor'd, to make the Terms homogeneous or of the same Order. When the Operation is performed it will be found that when $\lambda = 2$, $K =$

$$\frac{q}{q-r}, k = \frac{1}{q-r}. \quad L = \frac{r}{r-q}, l = \frac{1}{r-q}.$$

$$\text{If } \lambda = 3. K = \frac{qq-1}{q-r \times q-s}, k = \frac{q}{q-r \times q-s}.$$

$$L = \frac{rr-1}{r-q \times r-s}, l = \frac{r}{r-q \times r-s}.$$

$$M = \frac{ss-1}{s-q \times s-r}, m = \frac{s}{s-q \times s-r}.$$

As in the last Example.

If $\lambda = 4$.

$$K = \frac{q^3 - 2q}{q-r \times q-s \times q-t}, k = \frac{qq-1}{q-r \times q-s \times q-t}$$

$$L = \frac{r^3 - 2r}{r-q \times r-s \times r-t}, l = \frac{rr-1}{r-q \times r-s \times r-t}$$

M =

$$M = \frac{s^3 - 2s}{s-q \times s-r \times s-t}, m = \frac{ss - 1}{s-q \times s-r \times s-t}.$$

$$N = \frac{t^3 - 2t}{t-q \times t-r \times t-s}, n = \frac{tt - 1}{t-q \times t-r \times t-s}.$$

If $\lambda = 5$.

$$K = \frac{q^4 - 3q^2 + 1}{q-r \times q-s \times q-t \times q-v}, k = \frac{q^3 - 2q}{q-r \times q-s \times q-t \times q-v}.$$

$$L = \frac{r^4 - 3r^2 + 1}{r-q \times r-s \times r-t \times r-v}, l = \frac{r^3 - 2r}{r-q \times r-s \times r-t \times r-v}.$$

$$M = \frac{s^4 - 3ss + 1}{s-q \times s-r \times s-t \times s-v}, m = \frac{s^3 - 2s}{s-q \times s-r \times s-t \times s-v}.$$

$$N = \frac{t^4 - 3tt + 1}{t-q \times t-r \times t-s \times t-v}, n = \frac{t^3 - 2t}{t-q \times t-r \times t-s \times t-v}.$$

$$P = \frac{v^4 - 3v^2 + 1}{v-q \times v-r \times v-s \times v-t}, p = \frac{v^3 - 2v}{v-q \times v-r \times v-s \times v-t}.$$

And the Law of the Progression is evident; in the Denominator 'tis plain: And the Numerator of any Quantity K is determined by this Series, $q^{\lambda-1} - \frac{\lambda-2}{1} q^{\lambda-3} + \frac{\lambda-3}{1} \times \frac{\lambda-4}{2} q^{\lambda-5} - \frac{\lambda-4}{1} \times \frac{\lambda-5}{2} \times \frac{\lambda-6}{3} q^{\lambda-7} + \text{Ec.}$

If any body would know the Reason why the given Trinomial Quantity is divided into as many simple Trinomials, as there are Unites in λ , as is done in this and the last Example; they may see it demonstrated in Cor. 3. Prop. 46. of my Elements of Trigonometry.

Ex. 15.

To transform the Fluxion $x\dot{x} + ay\dot{x} + y\dot{y} = 0$. Put $y = zx$, and $\dot{y} = z\dot{x} + x\dot{z}$, then the Fluxion becomes $x\dot{x} + azx\dot{x} + z^2x\dot{x} + x^2z\dot{z} = 0$, whence $\dot{x} =$

$$\frac{-xz\dot{z}}{zz + az + 1}, \text{ and } \frac{\dot{x}}{x} = \frac{-z\dot{z}}{zz + az + 1}.$$

F

Ex.

Ex. 16.

To transform $axy + \dot{x}\sqrt{xx+yy} = 0$. Put $x = y\sqrt{zz-1}$, then $\dot{x} = \frac{zy\dot{y} - y + yz\dot{z}}{\sqrt{zz-1}}$, and the

Fluxion becomes $ay\dot{y}\sqrt{zz-1} + \frac{zy\dot{y} - y + yz\dot{z}}{\sqrt{zz-1}} \times y\dot{z}$

$= 0$, then $az^2y\dot{y} - ay\dot{y} + z^3y\dot{y} - zy\dot{y} + y^2z^2\dot{z} = 0$, therefore

$$\dot{y} = \frac{-y^2z^2\dot{z}}{ay \times \sqrt{zz-1} + z^3 - z \times y} = \frac{-yz^2\dot{z}}{a \times \sqrt{zz-1} + z^3 - z}$$

$$\text{and } \frac{\dot{y}}{y} = \frac{-z^2\dot{z}}{a \times \sqrt{zz-1} + z \times \sqrt{zz-1}} = \frac{-z^2\dot{z}}{a + z \times \sqrt{zz-1}}$$

$$\text{therefore } \frac{\dot{y}}{y} + \frac{z^2\dot{z}}{z + a \times z + 1 \times z - 1} = 0. \text{ And}$$

the last Term may be further transform'd by putting

$$\frac{\dot{z}}{z + a \times z + 1 \times z - 1} = \frac{A\dot{z}}{z + a} + \frac{B\dot{z}}{z + 1} + \frac{C\dot{z}}{z - 1}$$

and finding A, B, C, as in Ex. 5.

Ex. 17.

Let $ay^3 = x\dot{x}y^2 + b\dot{x}y^2 - c\dot{x}^3$, be given; put $x = zy$, then $ay^3 = xzy^3 + bzy^3 - czy^3$, or $a = xz + b - cz$, whence $x = \frac{cz^3 - bz + a}{z} = cz^2 - b + \frac{a}{z}$

$$\text{and } \dot{x} = 2cz\dot{z} - \frac{a\dot{z}}{z^2} = zy\dot{z}, \text{ whence } \dot{y} = 2c\dot{z} - \frac{a\dot{z}}{z^3}, \text{ \&c.}$$

Ex. 18.

To transform the Fluxion $\frac{\dot{x}}{\sqrt[n]{ax^p + bx^n}}$

put $ax^p + bx^n = x^n z^n$, then $z^n - b = ax^{p-n}$, and Log

$z^n - b$

$\overline{z^n - b} = \text{Log. } a + \overline{p - n} \times \text{Log. } x.$ In Fluxions

$$\frac{n z^{n-1} \dot{z}}{z^n - b} = \frac{p - n}{x} \dot{x}, \text{ hence } \frac{\dot{z}}{\sqrt[n]{ax^p + bx^n}} = \frac{\dot{x}}{xz}$$

$$= \frac{n}{p - n} \times \frac{z^{n-2} \dot{z}}{z - b}.$$

Ex. 19.

To transform the Fluxion $\overline{vx\dot{x} - xx\dot{v}} \times \text{Fl. } \frac{v^2 \dot{x}}{x}$
 $= c\dot{x}$; let $v = x^n$, and $\dot{v} = nx^{n-1} \dot{x}$; then will
 $x^{n+1} \dot{x} - nx^{n+1} \dot{x} \times \frac{x^{2n}}{2n} = c\dot{x}$, or $\frac{1-n}{2n} x^{3n+1} \dot{x} =$
 $c\dot{x}$, then comparing the Indexes and Coefficients on
 both Sides, $3n+1=0$, and $\frac{1-n}{2n} = c$; which gives
 $n = -\frac{1}{3}$, and $c = -2$. Therefore $v = \frac{1}{\sqrt[3]{x}}.$

PROP. X.

An Equation being given containing the Fluxions of Quantities; to find the Fluents, either in simple Terms, or in a Series thereof proceeding ad infinitum.

RULES.

When the *Fluxions* are not of the same Order in all the Terms, supply the Defect by the Powers of some given Fluxion supposed to be Unity: *Fractional* and *Compound Quantities* must be reduced by Multiplication, Division, &c. Radical Quantities (except such where the fluxionary Part is in a given Ratio to the

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Fluxion

Fluxion of the Root) must be *reduced* to simple Terms by *Involution*; and the Roots of *adseſſed Equations* must be *extracted*. This Preparation being made,

I.

1. If the Equation then can be so order'd, that every Term has only one variable Quantity and its Fluxion; multiply separately each Term by its variable Quantity, and then divide it by the Fluxion and its new Index.

2. If any Term be such a Radical Quantity that the fluxionary Part may be divided by the Fluxion of the Root (or Part under the Vinculum); and that by such Division the Quotient may be a given Quantity. Multiply that Term by the said Root, and then divide by the Fluxion of the Root and the new Index, for the Fluent of that radical Quantity.

3. If any Term be divided by the first Power of the variable Quantity; then the Fluent of that Term must be found by itself thus; multiply the given Coefficient and the Number 2.302585 into the Logarithm of that variable Quantity, for the Fluent of that Term. Or thus by Series, substitute for this variable Quantity the Sum or Difference of some given Quantity and another variable Quantity, and its Fluxion for the Fluxion; then this new Term being actually divided, the Fluent will be found as in the first Article.

4. If in one Side of an Equation there be two Terms containing two variable Quantities, each multiply'd into the Fluxion of the other; then by Art. 1. find the Fluent of either Term, supposing only one Quantity variable, and this will be the Fluent of both these Terms. And if there be three Terms containing three variable Quantities, where the Fluxion of each is multiply'd into the Product of the other two; then find the Fluent of any one of these Terms (by Art. 1.) considering the other Quantities as given; and this will be the Fluent of all three.

5. Lastly,

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5. Lastly, all the *Terms* being *collected* on the corresponding Sides of the Equation, will give an Equation containing the *Relation* of the variable Quantities.

EXAMPLE I.

Let $\dot{y} = dx - \dot{x}$, then the Fluent $y = dx - x$.

Ex. 2.

Let $\dot{x} = ay^2 \dot{y}$, the Fluent is $x = \frac{ay^3}{3}$.

Ex. 3.

Let $\dot{y} = \frac{-x\dot{x}}{\sqrt{aa - xx}}$, or $\dot{y} = \overline{aa - xx}^{-\frac{1}{2}} \times -x\dot{x}$; multiply by the Root $aa - xx$, and there is $\overline{aa - xx}^{\frac{1}{2}} \times -x\dot{x}$; divide this by $-2x\dot{x} \times \frac{1}{2}$, and there arises $\overline{aa - xx}^{\frac{1}{2}}$ or $\sqrt{aa - xx} = y$.

Ex. 4.

Suppose $\dot{y} = \frac{a\dot{x}}{x}$, then $y = 2.302585a \times \text{Log. } x$.
Or thus, let $b + z = x$, and $\dot{z} = \dot{x}$, then will $\dot{y} = \frac{a\dot{z}}{b+z} = \frac{a\dot{z}}{b} - \frac{az\dot{z}}{bb} + \frac{az^2\dot{z}}{b^3} - \frac{az^3\dot{z}}{b^4} + \&c.$
and the Fluent $y = \frac{az}{b} - \frac{az^2}{2b^2} + \frac{az^3}{3b^3} - \frac{az^4}{4b^4} + \&c.$

Ex. 5.

Let $\dot{z} = 2ay^3x\dot{x} + 3ax^2y^2\dot{y}$, where there are the Quantities y^3 and ax^2 , and one multiply'd into the other's Fluxion; the Fluent of $2ay^3x\dot{x}$ (supposing y invariable) is ay^3x^2 , therefore $z = ay^3x^2$.

Ex. 6.

$\dot{v} = ax^m\dot{x}$, then $v = \frac{ax^{m+1}}{m+1}$.

Ex.

Ex. 7.

Suppose $\dot{y} = \overline{ax + xx} \sqrt{2ax + xx}$, then $y =$

$$\frac{ax + xx \times 2ax + xx^{\frac{3}{2}}}{\frac{3}{2} \times 2ax + 2xx} = \frac{2ax + xx^{\frac{3}{2}}}{3}.$$

Ex. 8.

Let $\dot{v} = \frac{\dot{x}}{y} - \frac{x\dot{y}}{yy}$, that is $\dot{v} = y^{-1}\dot{x} - xy^{-2}\dot{y}$,
 and the Fluent $v = xy^{-1} = \frac{x}{y}.$

Ex. 9.

Let $\dot{v} = y^2z^3\dot{x} + 2xz^3y\dot{y} + 3xy^2z^2\dot{z}$; then $v =$
 $xy^2z^3.$

Ex. 10.

Suppose $\dot{y} = \frac{\dot{x}}{xx^3} - \frac{\dot{x}}{xx^2} + \frac{ax\dot{x}}{xx^{\frac{1}{2}}} - x^{\frac{1}{2}}\dot{x} + x^{\frac{3}{2}}\dot{x}$;
 multiply the first Side by $\frac{y}{\dot{y}}$, and the last by $\frac{x}{\dot{x}}$;
 and there will be $y = \frac{1}{xx} - \frac{1}{x} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}}$;
 then divide by the Index of the Power in each Term,
 and the Fluent will be $y = \frac{-1}{2xx} + \frac{1}{x} + 2ax^{\frac{1}{2}} -$
 $\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}.$

Ex. 11.

Let the given Equation be $\dot{x} = \frac{2bbc\dot{y}}{\sqrt{ay^3}} + \frac{3y^2\dot{y}}{a+b}$
 $+ \dot{y}\sqrt{by+cy}$, that is $\dot{x} = \frac{2bbc}{\sqrt{a}} \times y^{-\frac{3}{2}}\dot{y} + \frac{3}{a+b}$
 $\times y^2\dot{y} + y^{\frac{1}{2}}\dot{y}\sqrt{b+c}$; the Fluent is $x = \frac{-4bbc}{\sqrt{a}}y^{-\frac{1}{2}}$
 $+ \frac{y^3}{a+b} + \frac{2}{3}y^{\frac{3}{2}}\sqrt{b+c}.$

Ex.

Ex. 12.

Let $\dot{y} = x^{-\frac{1}{2}}\dot{x} - x^{\frac{1}{2}}\dot{x} + \frac{\dot{x}}{x}$; the Fluent will be y
 $= 2x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + 2.302585 \text{ Log. } x$. Or thus, put $a =$
 $z = x$, and $-\dot{z} = \dot{x}$, then $\frac{\dot{x}}{x} = \frac{-\dot{z}}{a-z} = -\frac{\dot{z}}{a} - \frac{z\dot{z}}{aa}$
 $-\frac{z^2\dot{z}}{a^3} - \mathcal{E}c$. whence $\dot{y} = x^{-\frac{1}{2}}\dot{x} - x^{\frac{1}{2}}\dot{x} - \frac{\dot{z}}{a} - \frac{z\dot{z}}{a^2}$
 $-\frac{z^2\dot{z}}{a^3} - \mathcal{E}c$. and then $y = 2x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} - \frac{z}{a} - \frac{z^2}{2aa}$
 $-\frac{z^3}{3a^3} - \mathcal{E}c$.

Ex. 13.

Let the Equation be $\dot{y} = \overline{\alpha + \beta z^\eta}^\mu z^\pi \dot{z}$; involve
 that radical Quantity, and then $\dot{y} = \alpha^\mu z^\pi \dot{z} +$
 $\mu \alpha^{\mu-1} \beta z^\eta + \pi \dot{z} + \mu \times \frac{\mu-1}{2} \alpha^{\mu-2} \beta^2 z^{2\eta} + \pi \dot{z} + \mathcal{E}c$.
 whence the Fluent is $y = \frac{\alpha^\mu z^{\pi+1}}{\pi+1} + \frac{\mu \alpha^{\mu-1} \beta z^{\eta+\pi+1}}{\eta+\pi+1}$
 $+ \mu \times \frac{\mu-1}{2} \times \frac{\alpha^{\mu-2} \beta^2 z^{2\eta+\pi+1}}{2\eta+\pi+1} + \mathcal{E}c$.

Ex. 14.

Suppose $\dot{y} = \theta e z^{\theta-1} \dot{z} + \theta f z^{\theta+\eta-1} \dot{z}$ into $\overline{e + f z^\eta}^{\mu-1}$,
 $+ \mu \eta f$
 that is $\dot{y} = \theta z^{\theta-1} \dot{z} \times \overline{e + f z^\eta}^\mu + \mu \eta f z^{\theta+\eta-1} \dot{z} \times \overline{e + f z^\eta}^{\mu-1}$.
 Here are the two Quantities z^θ and $\overline{e + f z^\eta}^\mu$ each mul-
 tiply'd into the other's Fluxion; therefore $y = z^\theta \times$
 $\overline{e + f z^\eta}^\mu$.

Ex.

Ex. 15.

Let $\dot{y} = \theta e z^{\theta-1} \dot{z} + \theta f z^{\theta+n-1} \dot{z} + \theta g z^{\theta+2n-1} \dot{z}$ into
 $+ \mu n f \quad + 2 \mu n g$

$\overline{e + f z^n + g z^{2n}}^{\mu-1}$, that is $\dot{y} = \theta z^{\theta-1} \dot{z} \times \overline{e + f z^n + g z^{2n}}^{\mu}$
 $+ \mu n f z^{\theta+n-1} \dot{z} + 2 \mu n g z^{\theta+2n-1} \dot{z} \times \overline{e + f z^n + g z^{2n}}^{\mu-1}$

Here are two Quantities z^{θ} and $\overline{e + f z^n + g z^{2n}}^{\mu}$ each multiply'd into the Fluxion of the other, therefore $y = z^{\theta} \times \overline{e + f z^n + g z^{2n}}^{\mu}$, the Fluent.

Ex. 16.

Suppose $\dot{y} = \theta e g z^{\theta-1} \dot{z} + \overline{\theta + \mu n} \times f g z^{\theta+n-1} \dot{z} +$
 $+ \overline{\theta + r n} \times e b$

$\overline{\theta + \mu n + r n} \times f b z^{\theta+2n-1} \dot{z}$ into $\overline{e + f z^n}^{\mu-1} \times \overline{g + b z^n}^{r-1}$

that is $\dot{y} = \theta z^{\theta-1} \dot{z} \times \overline{e + f z^n}^{\mu} \times \overline{g + b z^n}^r + \mu n f z^{\theta+n-1} \dot{z}$
 $\times \overline{g + b z^n}^r \cdot \overline{e + f z^n}^{\mu-1} + r n b z^{\theta+2n-1} \dot{z} \cdot \overline{e + f z^n}^{\mu} \cdot \overline{g + b z^n}^{r-1}$

Now here are the 3 Quantities z^{θ} , $\overline{e + f z^n}^{\mu}$ and $\overline{g + b z^n}^r$, and the Fluxion of each is multiply'd into the Product of the rest; therefore the Fluent is $y = z^{\theta} \times \overline{e + f z^n}^{\mu} \times \overline{g + b z^n}^r$.

Ex. 17.

Let this Equation be given $\dot{y}^2 = \dot{x} \dot{y} + x^2 \dot{x}^2$; by extracting the Root, $\dot{y} = \frac{1}{2} \dot{x} \pm \dot{x} \sqrt{\frac{1}{4} + x}$ = (by Involution) $\frac{1}{2} \dot{x} + \frac{1}{2} \dot{x} + x^2 \dot{x} - x^4 \dot{x} + 2 x^6 \dot{x}$, &c. or $\dot{y} = \frac{1}{2} \dot{x} - \frac{1}{2} \dot{x} - x^2 \dot{x} + x^4 \dot{x} - 2 x^6 \dot{x}$, &c. Whence $y = x +$

$$\frac{x^3}{3} - \frac{x^5}{5} + \frac{2x^7}{7} \text{ \&c. or } y = \frac{-x^3}{3} + \frac{x^5}{5} - \frac{2x^7}{7} \text{ \&c.}$$

Ex

Ex. 18.

Suppose $\dot{y}^3 + ax\dot{x}^2\dot{y} + a^2\dot{x}^2\dot{y} - x^3\dot{x}^3 - 2a^3\dot{x}^3 = 0$. Extracting the Root of this Equation, $\dot{y} = ax - \frac{x\dot{x}}{4} + \frac{x^2\dot{x}}{64a}$
 $+ \frac{131x^3\dot{x}}{512a^2}$, &c. then the Fluent will be $y = ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2} + \&c.$

Ex. 19.

Suppose this Equation $\dot{z}^3 - c\dot{z}^2 - 2x^2\dot{z} - c^2\dot{z} + 2x^3 + c^3 = 0$. Put $\dot{x} = 1$, by which multiply the deficient Terms, and then you have $\dot{z}^3 - c\dot{z}^2\dot{x} - 2x^2\dot{x}^2\dot{z} - c^2\dot{x}^2\dot{z} + 2x^3\dot{x}^3 + c^3\dot{x}^3 = 0$. And extracting the Root, which will be threefold, then $\dot{z} = c\dot{x} + x\dot{x} - \frac{x^2\dot{x}}{4c} + \frac{x^3\dot{x}}{32cc}$, &c. Or $\dot{z} = c\dot{x} - x\dot{x} + \frac{3x^2\dot{x}}{4c} + \frac{15x^3\dot{x}}{32cc}$, &c. Or, lastly, $\dot{z} = -c\dot{x} - \frac{x^2\dot{x}}{2c} - \frac{x^3\dot{x}}{2cc} + \frac{x^5\dot{x}}{4c^4}$, &c. Whence the Fluent will either be $z = cx + \frac{1}{2}xx - \frac{x^3}{12c} + \frac{x^4}{128cc}$, &c. or $z = cx - \frac{1}{2}x^2 + \frac{x^3}{4c} + \frac{15x^4}{128cc}$, &c. or $z = -cx - \frac{x^3}{6c} - \frac{x^4}{8cc} + \frac{x^6}{24c^4}$, &c.

Ex. 20.

Let $\frac{a^4\dot{y}^2}{y} - 4a^3\dot{y}^2 + 4aay\dot{y}^2 + \frac{aaxx}{y}\dot{y}^2 = 4ax^2\dot{y}^2 + \frac{x^2\dot{x}^2}{y} - 4x^2y\dot{y}^2$; transpose $4ax^2\dot{y}^2 - 4x^2y\dot{y}^2$ and then multiply by y , and you have $a^4\dot{y}^2 - 4a^3y\dot{y}^2 + 4a^2y^2\dot{y}^2 + aax^2\dot{y}^2 - 4ax^2y\dot{y}^2 + 4y^2x^2\dot{y}^2 = x^2\dot{x}^2$; divide
G by

by $aa + xx$, and there arises $aa\dot{y}^2 - 4a\dot{y}\dot{y}^2 + 4\dot{y}^2\dot{y}^2 = \frac{x^2\dot{x}^2}{aa + xx}$, extract the Root, and $a\dot{y} - 2\dot{y}\dot{y} = \frac{x\dot{x}}{\sqrt{aa + xx}}$; whence the Fluent is $a\dot{y} - \dot{y}\dot{y} = \pm \frac{\sqrt{aa + xx}}{\sqrt{aa + xx}}$.

Ex. 21.

Let $\dot{x}\dot{y} + z\dot{y} = a\ddot{x}$; then its Fluent is $z\dot{y} = a\dot{x}$.

Ex. 22.

Suppose $a\dot{y}\dot{x}^2 - b\dot{y}\dot{x}^2 = bxx\dot{y} + b\dot{x}^3$, or $a\dot{y}\dot{x}^2 - b\dot{y}\dot{x}^2 - bxx\dot{y} = b\dot{x}^3$, supposing $\dot{x}$ invariable; then its Fluent will be $a\dot{y}\dot{x}^2 - bxx\dot{y} = bxx\dot{x}^2$.

Ex. 23.

Let the Equation be $az\dot{z}^2 + aaz\dot{y} - ab\dot{z}\dot{x} = bcz\dot{z} - abc\dot{x}$; then the Fluent is $\frac{az^2\dot{z}}{2} + aay\dot{z} - abx\dot{z} = bcz\dot{z} - abc\dot{x}$; supposing $\dot{z}$ invariable.

II.

If the Quantities cannot be so separated but one or both of the *Fluxions* contained in the Equation will be affected with *both* the *variable Quantities*; then reduce the Equation so that one of the Fluxions alone, (or at most, only affected with its own flowing Quantity) may possess one Side of the Equation, and the other Fluxion affected with both the variable Quantities be on the other Side, in simple Terms. Then

1. Range all the Quantities that are on the second Side of the Equation so, that all the fluxional Quantities affected only with its *own Fluent*, may stand *horizontally* at Top, proceeding regularly according to the Indices either increasing or decreasing, according

According as the Fluent is to be had in an ascending or descending Series; and all the Terms affected with the other flowing Quantity may stand *perpendicularly* on the *left Hand* according to their Indices.

2. Begin at the left Hand, and find the Fluent of the *first* Term of the *horizontal* Row, for the first Term of the Fluent; then substitute this instead of the other variable Quantity in all the Terms of the *perpendicular* Row, writing their new Values over-against them under their proper Indices in the *horizontal* Row, then proceed to find the second, third, &c. Term of the Fluent; by summing up all the fluxionary Quantities of the same Index into one Term, and then finding its Fluent; all which Terms of the Fluent are to be gradually substituted for the Powers of the other Quantity as you go along.

3. And this Operation may be performed various Ways, by *assuming* any given Quantity for the first Term, or perhaps for some other Term of the Fluent, and oftentimes it will be *necessary* to do so. If any of the Terms be divided by the *first Power* of its flowing Quantity, it will sometimes be necessary to substitute for this Quantity, the Sum or Difference of a given Quantity and another variable Quantity; and then reduce the Terms to the prescribed Form. And this is the *Newtonian Rule* for finding the *Fluent*.

Ex. 24.

Let $ay + xy - ax - xx = y\dot{x} + \frac{ay^3\dot{x} + xy^3\dot{x}}{a^3} - \frac{ax^2\dot{x} + x^3\dot{x}}{a^2}$, to find y . Divide by $a+x$ and reduce

$$\text{the Equation, then } y = \dot{x} + \frac{y\dot{x}}{a+x} + \frac{y^3\dot{x}}{a^3} - \frac{x^2\dot{x}}{a^2} \\ = \dot{x} - \frac{x^2\dot{x}}{aa} + \frac{y^3\dot{x}}{a^3} + \frac{y\dot{x}}{a} - \frac{yxx\dot{x}}{aa} + \frac{yx^2\dot{x}}{a^3}$$

&c. Then the Work will be as follows.

	$\dot{x}$	*	$-\frac{x^2\dot{x}}{aa}$	
$+$	$\frac{y\dot{x}}{a}$	$+$	$\frac{x\dot{x}}{a}$	$+\frac{x^2\dot{x}}{2aa}-\frac{x^3\dot{x}}{2a^3}, \text{ \Ô.}$
$-$	$\frac{yx\dot{x}}{aa}$		$-\frac{x^2\dot{x}}{aa}$	$-\frac{x^3\dot{x}}{2a^3}, \text{ \Ô.}$
$+$	$\frac{yx^2\dot{x}}{a^3}$			$+\frac{x^3\dot{x}}{a^3}, \text{ \Ô.}$
	$\&c.$			
$+$	$\frac{y^3\dot{x}}{a^3}$			$+\frac{x^3\dot{x}}{a^3}, \text{ \Ô.}$
$y =$	$\dot{x}$	$+$	$\frac{x\dot{x}}{a}$	$-\frac{3x^2\dot{x}}{2aa} + \frac{x^3\dot{x}}{a^3} \text{ \Ô.}$
$y =$	x	$+$	$\frac{x^2}{2a}$	$-\frac{x^3}{2aa} + \frac{x^4}{4a^3} \text{ \Ô.} = \text{Fluent.}$

In this Example I write $\dot{x} - \frac{x^2\dot{x}}{aa}$ horizontally, and
 $\frac{y\dot{x}}{a} - \frac{yx\dot{x}}{aa} + \frac{yx^2\dot{x}}{a^3} \text{ \Ô.} + \frac{y^3\dot{x}}{a^3}$ perpendicular-
 larly; then I bring down $\dot{x}$ into the Value of y , and get
 its Fluent x , and put it into the Value of y . Then I
 substitute x for y in each Term of the perpendicular
 Row, placing their respective Values $\frac{x\dot{x}}{a}$, $-\frac{x^2\dot{x}}{aa}$,
 $+\frac{x^3\dot{x}}{a^3}$, and $+\frac{x^3\dot{x}}{a^3}$ against each, under the pro-
 per Powers of x .

Then I bring down the Term in the second Place
 $+\frac{x\dot{x}}{a}$, and write its Fluent $\frac{x^2}{2a}$ in the Value of
 y ; then I write $\frac{x^2}{2a}$ for y in the Terms $+\frac{y\dot{x}}{a} -$
 $\frac{yx\dot{x}}{aa}, \text{ \Ô.}$ and the Result $\frac{x^2\dot{x}}{2a^2}, -\frac{x^3\dot{x}}{2a^3}$ I put over-
 against

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against them as before. Then I take the Sum of the Terms in the third Place — $\frac{x^2\dot{x}}{aa} + \frac{x^2\dot{x}}{2aa} - \frac{x^2\dot{x}}{aa}$ and set this Sum — $\frac{3x^2\dot{x}}{2aa}$ underneath, and get its Fluent — $\frac{x^3}{2aa}$, which I substitute for y as before. And thus I proceed as far as I please.

Ex. 25.

Let $\dot{y} = \dot{x} - 3x\dot{x} + y\dot{x} + x^2\dot{x} + yx\dot{x}$, to find y .

Here x the first Term being found and written for y , you get — $2x\dot{x}$, and its Fluent — xx for the second Term, which being also written for y , you'll get + $x^2\dot{x}$, and its Fluent $\frac{x^3}{3}$ for the third Term; and so on as below.

		$+ \dot{x} - 3x\dot{x} + x^2\dot{x}$				
$+ y\dot{x}$		$+ x\dot{x} - x^2\dot{x} + \frac{1}{3}x^3\dot{x} - \frac{1}{6}x^4\dot{x} \quad \mathcal{E}c.$				
$+ yx\dot{x}$		$+ x^2\dot{x} - x^3\dot{x} + \frac{1}{3}x^4\dot{x} \quad \mathcal{E}c.$				
$\dot{y} = \dot{x}$		—	$2x\dot{x}$	+	$x^2\dot{x}$	— $\frac{2}{3}x^3\dot{x} + \frac{1}{6}x^4\dot{x} \quad \mathcal{E}c.$
$y = x$		—	xx	+	$\frac{1}{3}x^3$	— $\frac{1}{6}x^4 + \frac{1}{30}x^5 \quad \mathcal{E}c.$

Otherwise thus.

Here I take a for the first Term of y , and writing it instead of y , and get $ax + x$ the next Term, which being again written for y , I get the third Term $ax^2 - x^2$, and so on; see the Work.

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	$+ \dot{x} - 3x\dot{x} + x^2\ddot{x}$
$+ y\dot{x}$	$a\dot{x} + \frac{a}{1}x\dot{x} + \frac{a}{1}x^2\ddot{x} + \frac{\frac{2}{3}a}{+\frac{1}{3}}x^3\ddot{x} \&c.$
$+ yx\dot{x}$	$+ ax\dot{x} + \frac{a}{1}x^2\ddot{x} + \frac{a}{1}x^3\ddot{x} \&c.$
$\dot{y} =$	$+\frac{a}{1}\dot{x} + \frac{2a}{-2}x\dot{x} + \frac{2a}{1}x^2\ddot{x} + \frac{\frac{5}{3}a}{-\frac{2}{3}}x^3\ddot{x} \&c.$
$y = a +$	$\frac{a}{1}x + \frac{a}{-1}x^2 + \frac{\frac{2}{3}a}{+\frac{1}{3}}x^3 + \frac{\frac{5}{12}a}{-\frac{1}{6}}x^4 \&c.$

Ex. 26.

Suppose this Equation $\dot{y} = -3x\dot{x} + 3yx\dot{x} + y^2\ddot{x} - y^2x\ddot{x} + y^3\ddot{x} - y^3x\ddot{x} \&c. + 6yx^2\ddot{x} - 6x^2\ddot{x} + 8yx^3\ddot{x} - 8x^3\ddot{x} + 10yx^4\ddot{x} - 10x^4\ddot{x} \&c.$ To find the Value of y as far as 7 Dimensions of x .

I place the Terms in Order according to the following Table, and then I work as before; and moreover I subjoin the Square and Cube of the Value of y gradually produced, to be substituted by Degrees into their proper Places towards the right Hand, in the Values of the Marginals on the left, as follows.

	$-3x\dot{x} - 6x^2\dot{x} - 8x^3\dot{x} - 10x^4\dot{x} - 12x^5\dot{x} - 14x^6\dot{x}$	$\mathcal{E}c.$
$+ 3y\dot{x}\dot{x}$	$- \frac{5}{2}x^3\dot{x} - 6x^4\dot{x} - \frac{75}{8}x^5\dot{x} - \frac{273}{20}x^6\dot{x}$	$\mathcal{E}c.$
$+ 6y\dot{x}^2\dot{x}$	$- 9x^4\dot{x} - 12x^5\dot{x} - \frac{75}{4}x^6\dot{x}$	$\mathcal{E}c.$
$+ 8y\dot{x}^3\dot{x}$	$- 12x^5\dot{x} - 16x^6\dot{x}$	$\mathcal{E}c.$
$+ 10y\dot{x}^4\dot{x}$	$- 15x^6\dot{x}$	$\mathcal{E}c.$
$\mathcal{E}c.$		
$+ y\dot{y}\dot{x}$	$+ \frac{9}{4}x^4\dot{x} + 6x^5\dot{x} + \frac{107}{8}x^6\dot{x}$	$\mathcal{E}c.$
$- y\dot{x}\dot{x}$	$- \frac{9}{4}x^5\dot{x} - 6x^6\dot{x}$	$\mathcal{E}c.$
$\mathcal{E}c.$		
$+ y^3\dot{x}$	$- \frac{27}{8}x^6\dot{x}$	$\mathcal{E}c.$
$y = -3x\dot{x} - 6x^2\dot{x} - \frac{25}{2}x^3\dot{x} - \frac{91}{4}x^4\dot{x} - \frac{333}{8}x^5\dot{x} - \frac{367}{5}x^6\dot{x}$	$\mathcal{E}c.$	
$y = -\frac{3}{2}x^2 - 2x^3 - \frac{25}{8}x^4 - \frac{91}{20}x^5 - \frac{111}{16}x^6 - \frac{367}{35}x^7$	$\mathcal{E}c.$	
$y^2 = \frac{9}{4}x^4 + 6x^5 + \frac{107}{8}x^6$	$\mathcal{E}c.$	
$y^3 = -\frac{27}{8}x^6$	$\mathcal{E}c.$	

Ex. 27.

Let $\dot{y} = \frac{y\dot{x}}{xx} + \frac{\dot{x}}{xx} + 3\dot{x} + 2x\dot{x} - \frac{4\dot{x}}{x}$; to find

y in a descending Series. Here the Terms in the horizontal Row must be placed to proceed from the greater Indices to the lesser: And the Work will be as below.

	$+ 2x\dot{x} + 3\dot{x} - \frac{4\dot{x}}{x} + \frac{\dot{x}}{xx}$
$+ \frac{y\dot{x}}{xx}$	$\dot{x} + \frac{4\dot{x}}{x} * - \frac{\dot{x}}{x^3} + \frac{\dot{x}}{2x^4} \&c.$
$\dot{y} = 2x\dot{x} + 4\dot{x} + 0 + \frac{\dot{x}}{xx} - \frac{\dot{x}}{x^3} + \frac{\dot{x}}{2x^4}$	$\&c.$
$y = x^2 + 4x * - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3}$	$\&c.$

Here

Here observe, that any given Quantity might have been inserted between the Terms $4x$ and $-\frac{1}{x}$; and so y might be extracted an infinite Variety of Ways. But if that fluxionary Term had not vanished, then we had been obliged to substitute $a+z$ or $a-z$ for x in the given Equation before it could be resolved.

Ex. 28.

Suppose the given Equation $cx^2\dot{x} + y\dot{x} = a\dot{y}$, by Reduction $\dot{y} = \frac{cx^2\dot{x}}{a} + \frac{y\dot{x}}{a}$; then the Work will be very easily performed as in the following Table.

	$\frac{cx^2\dot{x}}{a}$
$+\frac{y\dot{x}}{a}$	$+\frac{cx^3\dot{x}}{3aa} + \frac{cx^4\dot{x}}{3.4a^3} + \frac{cx^5\dot{x}}{3.4.5a^4}$
$\dot{y} =$	$\frac{cx^2\dot{x}}{a} + \frac{cx^3\dot{x}}{3aa} + \frac{cx^4\dot{x}}{3.4a^3} + \frac{cx^5\dot{x}}{3.4.5a^4}$
$y =$	$\frac{cx^3}{3a} + \frac{cx^4}{3.4aa} + \frac{cx^5}{3.4.5a^3} + \frac{cx^6}{3.4.5.6a^4} \&c.$

Therefore $y = 2caa \times \frac{x^3}{2.3a^3} + \frac{x^4}{2.3.4a^4} + \frac{x^5}{2.3.4.5a^5} \&c.$ But (by Sch. II. Prob. 2. Sect. II.) $1 + \frac{x}{a} + \frac{x^2}{2aa} + \frac{x^3}{2.3a^3} + \frac{x^4}{2.3.4a^4} \&c. =$ Number of the Hyp. Log. $\frac{x}{a}$. Therefore $y = 2caa \times$ Num

of the Hyp. Log. $\frac{x}{a} - 2caa \times 1 + \frac{x}{a} + \frac{xx}{2aa}$

And by the same Rule the Fluent of $x^2\dot{y} = x\dot{x} - yx^2\dot{x}$, will be found $y = \frac{1}{x}$.

III.

When the given Equation contains first Fluxions alone, or if it contains *first, second, or third* &c. Fluxions, as in the following Example, where $2axy^2 - ay^2\dot{x} + 2x^2\dot{y}^2 - 2y^2\dot{x}^2 = 0$, to find x expressed by y and given Quantities; it will be resolved by the following general Method.

1. Make the Equation $= 0$; and assume an *inde-termin'd Series* to represent the Series required; as $x = Ay^n + By^{n+r} + Cy^{n+s} + Dy^{n+t}$, &c. wherein the Indices $n, n+r, n+s$ continually increase if y be very small, or decrease if it be great.

2. To find the first Index n ; substitute into the given Equation the *first Term* Ay^n , its *Fluxion*, and *second Fluxion*, &c. instead of $x, \dot{x}$, and $\ddot{x}$, &c. (if they be there) and then you'll have a new Equation, as $2aAy^n - \overline{nn-n} \times aAy^n + 2A^2y^{2n} - 2n^2A^2y^{2n} = 0$, supposing $\dot{y} = 1$. Make two (or more) of the Terms equal to nothing that have the *least Indices* equal to one another, for an *ascending Series*; or those that have the *greatest Indices* equal, for a *descending Series*; then by equating their Indices or else their Coefficients, n will be found. Or if there happen to be only *one Term* with such least or greatest Index, make its *Coefficient* $= 0$, which will destroy that Term, and perhaps give the Value of n : Thus, in this Example for an ascending Series, you'll have $2aAy^n - \overline{nn-n} \times aAy^n = 0$, or $\overline{nn-n} = 2$, whence $n = 2$.

3. For the other Indices; substitute the Value of n into the foregoing Equation, and you'll have $2aAy^2 - 2aAy^2 + 2A^2y^4 - 8A^2y^4 = 0$. Then take the *least Index* in an *ascending Series*, or the *greatest* in a *descending* one from each of the rest; and find all the

H possible

possible Numbers that result by *adding* all these Remainders to themselves and to one another as oft as possible, and then you have $r, s, t, \&c.$ here $4 - 2 = 2$ only one Remainder; then $2, 4, 6, \&c. = r, s, t, \&c.$ And therefore the Series is $Ay^2 + By^4 + Cy^6 + Dy^8 \&c. = x.$

4. For determining the Coefficients $A, B, C, \&c.$ substitute into the given Equation the *Values* of $x, \dot{x}, \ddot{x}, \&c.$ expressed by the foregoing Series $Ay^2 + By^4 + Cy^6, \&c.$ and put the *Sum* of the *Coefficients* of every several Power of y equal to nothing, and thence $A, B, C, \&c.$ will be gradually found. Thus $\dot{x} = 2Ay + 4By^3 + 6Cy^5, \&c.$ $\ddot{x} = 2A + 12By^2 + 30Cy^4, \&c.$ when $y=1$, these being substituted, the Work will be performed as below.

5. If the first Equation for the Coefficients be an *affected* Equation containing several *different Powers* of A , then that Equation will afford several Roots or Values of A ; and as many different Roots so many different Series may be obtain'd. And if A has several equal Values or Roots in this Equation, then you must divide the least Remainder above by that Number (of equal Roots of A , one of which you assume for its Value;) then proceed as in Art. 3. taking this Quotient for another Remainder.

6. If a Series be required to be express'd in Terms of that *Quantity* whose 2d, 3d Fluxion, $\&c.$ is in the Equation; it must first be got in Terms of the *other* Quantity that has no second, third, $\&c.$ Fluxion and then the Series reverted.

SCHOL. As there are fluxionary Equations that admit of several Solutions, and may have the Root express'd various ways; so there are Equations that cannot be resolved at all as being impossible; and others that are very difficult to be resolved, and may require the utmost Skill of the Analyst, and sometimes a different Process from these Rules.

Ex. 29.

Let $2axy^2 - ay^2\dot{x} + 2x^2\dot{y}^2 - 2y^2\dot{x}^2 = 0$; to find x expressed by y and given Quantities.

The Form of the Series found by the foregoing Rule is $x = Ay^2 + By^4 + Cy^6 + Dy^8 \&c.$ This and its Fluxions being substituted into the given Equation will be as follows.

$$\begin{array}{rcl}
 + 2axy^2 & + 2aAy^2 + 2aBy^4 + 2aCy^6 + 2aDy^8 \&c. \\
 - ay^2\dot{x} & - 2aAy^2 - 12aBy^4 - 30aCy^6 - 56aDy^8 \&c. \\
 + 2x^2\dot{y}^2 & + 2A^2y^4 + 4ABy^6 + 2BB + 4ACy^8 \&c. \\
 - 2y^2\dot{x}^2 & - 8A^2y^4 - 32ABy^6 - 32BB - 48ACy^8 \&c.
 \end{array}$$

Then equating their respective Coefficients; $2aA - 2aA = 0$, therefore A may be taken at Pleasure.

Again $-10aB - 6A^2 = 0$, thence $B = \frac{-3A^2}{5a}$.

After the same Manner $C = \frac{3A^3}{5aa}$, and $D = -\frac{31A^4}{45a^3}$.

$\&c.$ Whence $x = Ay^2 - \frac{3A^2}{5a}y^4 + \frac{3A^3}{5aa}y^6 - \frac{31A^4}{45a^3}y^8 \&c.$

Otherwise thus for a descending Series:

Make the Terms $2A^2y^{2n} - 2n^2A^2y^{2n} = 0$, or $1 - nn = 0$, whence $n = 1$; this substituted for n will produce $2aAy + 2A^2y^2 - 2A^2y^2 = 0$. Take the greatest Index 2 from the rest, which is 1, and there remains -1 ; therefore $r, s, t, \&c.$ are $-1, -2, -3, \&c.$ and the Series is $Ay + B + Cy^{-1} + Dy^{-2} \&c.$ and the Operation will be as follows.

$$\begin{array}{r}
 + 2axy\dot{y}^2 \quad \left| \quad + 2aAy + 2aB + 2aCy^{-1} \quad \mathcal{E}c. \\
 - ay^2\ddot{x} \quad \left| \quad - 2aCy^{-1} \quad \mathcal{E}c. \\
 + 2x^2\dot{y}^2 \quad \left| \quad + 2A^2y^2 + 4ABy + 2BB + 4BC \quad \mathcal{E}c. \\
 - 2y^2\dot{x}^2 \quad \left| \quad - 2A^2y^2 + 4AC + 4ADy^{-1} \quad \mathcal{E}c. \\
 \text{or } \left\{ \begin{array}{l} + 2A^2y^2 + 2aA + 2aB + 4BC \\ - 2A^2y^2 + 4ABy + 8AC + 12ADy^{-1} \end{array} \right. \mathcal{E}c. = 0.
 \end{array}$$

Hence, equating the Coefficients; $2A^2 - 2A^2 = 0$, and A may be again taken at Pleasure; likewise

$$B = \frac{-a}{2}, \quad C = \frac{aa}{16A}, \quad D = \frac{a^3}{96A^2}, \quad \mathcal{E}c. \text{ whence}$$

the Series is known, and $x = Ay - \frac{a}{2} + \frac{aa}{16Ay}$

$$+ \frac{a^3}{96A^2y^2}, \quad \mathcal{E}c.$$

Ex. 30.

Let $a^2\dot{y} - a^2y\ddot{y} + yy\dot{x}^2 = 0$, to find y in an ascending Series.

Assume $y = Ax^n + Bx^{n+r} + Cx^{n+s}$, $\mathcal{E}c.$ and substituting the first Term and its Fluxions for y and its Fluxions, we have $n \times n - 1 \times n - 2 \times a^2 Ax^{n-3} - n^2 \times n - 1 \times a^2 A^2 x^{2n-3} + nA^2 x^{2n-1} = 0$. Since the Index $n-3$ is the least Index, make its Coefficient $n \times n - 1 \times n - 2 = 0$, and take one of the Roots $n=2$. Whence the Indices will be $-1, 1, 3$, subtract the least, -1 from the rest, and there remains $2, 4$; therefore $r, s, t = 2, 4, 6$ $\mathcal{E}c.$ whence $y = Ax^2 + Bx^4 + Cx^6$ $\mathcal{E}c.$ and the rest of the Work is as follows.

$$\begin{array}{l|l} + a^3 \ddot{y} & + 24a^3 Bx + 120a^3 Cx^3 + 336a^3 Dx^5 \quad \mathcal{E}c. \\ - a^2 \dot{y}\dot{y} & - 4a^2 A^2 x - 32a^2 ABx^3 - 72a^2 ACx^5 \quad \mathcal{E}c. \\ + y\dot{y}\dot{x}^2 & - 48a^2 BB \\ & + 2A^2 x^3 + 6ABx^5 \quad \mathcal{E}c. \end{array}$$

Hence $24a^3 B = 4a^2 A^2$, and A may be taken at Pleasure; let $A = \frac{1}{2a}$, then $B = \frac{1}{24a^3}$, also $C = \frac{1}{720a^5}$, $D = \frac{1}{40320a^7}$ $\mathcal{E}c.$ Whence $y = \frac{x^2}{2a} + \frac{x^4}{24a^3} + \frac{x^6}{720a^5} + \frac{x^8}{40320a^7}$ $\mathcal{E}c.$

Otherwise for a descending Series.

Let $2n-1$ and $n-3$ be supposed to be the greatest Indices, then $n-3=2n-1$, and $n=-2$, and these Indices will be $-5, -7$; and taking -5 from -7 the Remainder is -2 ; therefore $r, s, t = -2, -4, -6$, $\mathcal{E}c.$ Wherefore $y = Ax^{-2} + Bx^{-4} + Cx^{-6}$, $\mathcal{E}c.$ the rest of the Work will be thus,

$$\begin{array}{l|l} + a^3 \ddot{y} & - 24a^3 Ax^{-5} - 120a^3 Bx^{-7} - 336a^3 Cx^{-9} \quad \mathcal{E}c. \\ - a^2 \dot{y}\dot{y} & \phantom{- 24a^3 Ax^{-5} - 120a^3 Bx^{-7} - 336a^3 Cx^{-9}} + 12a^2 A^2 x^{-7} + 64a^2 ABx^{-9} \quad \mathcal{E}c. \\ + y\dot{y}\dot{x}^2 & - 2A^2 x^{-5} - 6ABx^{-7} - 4BBx^{-9} \quad \mathcal{E}c. \end{array}$$

Therefore $A = -12a^3$, $B = 36a^5$, $C = -\frac{1368}{10}a^7$ $\mathcal{E}c.$

And $y = -12a^3 x^{-2} + 36a^5 x^{-4} - \frac{1368}{10}a^7 x^{-6}$ $\mathcal{E}c.$

Ex. 31.

Let the Equation be $\dot{x} = \frac{1}{2}y\dot{y} - 4y^2\dot{y} + 2x^{\frac{1}{2}}y\dot{y} - 3x^2\dot{y} + 7y^{\frac{5}{2}}\dot{y} + 2y^3\dot{y}$, to find x in y ascending.

Let

The DOCTRINE

Let $x = Ay^n + By^{n+r} + Cy^{n+s} \&c.$; and substituting Ay^n and its Fluxion for x and $\dot{x}$, there arises $-nAy^{n-1}$

$+ \frac{1}{2}y - 4y^2 + A^{\frac{1}{2}}y^{\frac{n}{2}+1} - \frac{4}{3}A^2y^{2n} + 7y^{\frac{5}{2}} + 2y^3 = 0$. Suppose $n-1$ and 1 to be the least Indices, and you will have $-nAy^{n-1} + \frac{1}{2}y = 0$, and $n-1=1$, or $n=2$ and all the Indices will be $1, 2, 4, 2\frac{1}{2}, 3$. Take from the Rest and the Remainders are $1, 1\frac{1}{2}, 2, 3$ whence $r, s, t \&c.$ are $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, \&c.$ and $x = Ay^2 + By^3 + Cy^{3\frac{1}{2}} + Dy^4, \&c.$ hence

$$\begin{array}{r|l}
 -\dot{x} & -2Ay - 3By^2 - 3\frac{1}{2}Cy^{2\frac{1}{2}} - 4Dy^3 \&c. \\
 + \frac{1}{2}y & + \frac{1}{2}y \\
 - 4y^2 & - 4y^2 \\
 + 2yx^{\frac{1}{2}} & + 2A^{\frac{1}{2}}y^{\frac{3}{2}} \quad * \quad + \frac{By^3}{A^{\frac{1}{2}}} \\
 - \frac{4}{3}x^2 & \quad * \quad * \quad * \\
 + 7y^{\frac{5}{2}} & \quad \quad + 7y^{\frac{5}{2}} \quad * \\
 + 2y^3 & \quad \quad \quad + 2y^3
 \end{array}$$

therefore $A = \frac{1}{4}$, likewise $B = -1$, $C=2$, $D=0 \&c.$ and the Series is $x = \frac{1}{4}y^2 - y^3 + 2y^{\frac{7}{2}}, \&c.$

Ex. 32.

Let $\dot{z}^3 - c\dot{z}^2 - 2x^2\dot{z} - c^2\dot{z} + 2x^3 + c^3 = 0$, to find z in a Series of x ascending. Put $z = Ax^n + Bx^{n+1} + Cx^{n+2} \&c.$ and by Substitution according to the Rule, we have $n^3A^3x^{3n-3} - cn^2A^2x^{2n-2} - 2nAx^{n+1} - nccAx^{n-1} + 2x^3 + c^3 = 0$; and supposing the least Indices $n-1$ and 0 to be equal, we have $n=1$. And all these Indices will become $0, 2, 3$: And subtracting the least 0 , the Remainders will be $2, 3$; whence $r, s, t, \&c. = 2, 3, 4, \&c.$ and the Series is $z = Ax + Bx^3 + Cx^4 + Dx^5 \&c.$ and the Operation follows,

+ z

$$\begin{array}{r|l}
 + \dot{z}^3 & + A^3 + 9A^2Bx^2 + 12A^2Cx^3 + 15A^2Dx^4 \quad \& c. \\
 & + 27AB^2x^4 \quad \& c. \\
 - c\dot{z}^2 & - cA^2 - 6cABx^2 - 8cACx^3 - 9cB^2x^4 \quad \& c. \\
 & - 10cADx^4 \quad \& c. \\
 - 2x^2\dot{z} & - 2Ax^2 \quad * \quad - 6Bx^4 \quad \& c. \\
 - c^2\dot{z} & - c^2A - 3c^2Bx^2 - 4c^2Cx^3 - 5c^2Dx^4 \quad \& c. \\
 + 2x^3 & + 2x^3 \quad * \\
 + c^3 & + c^3
 \end{array}$$

Here $A^3 - cA^2 - c^2A + c^3 = 0$, and $A = +c$ or $-c$: If $A = +c$, then B will be infinite; therefore $A = -c$, and then $B = -\frac{1}{6c}$, $C = -\frac{1}{8cc}$, $D = 0$, $\& c.$ And $z = -cx - \frac{x^3}{6c} - \frac{x^4}{8cc} \quad \& c.$

If you take $A = +c$; then since the Equation $A^3 - cA^2 - c^2A + c^3 = 0$, contains two Roots $= c$; therefore divide 2 (the least Remainder) by 2 (the Number of equal Roots) and the Quotient is 1. Then by Help of the Remainders 1, 2, 3, $\& c.$ you'll get the Series $z = Ax + Bx^2 + Cx^3, \& c.$ with this proceed as with the former, and you will get other Series for the Value of z , wherein B may be taken at Pleasure.

Ex. 33.

Suppose $ey + fzm\dot{y} + dyzm^{-1}\dot{z} = z^p\dot{z}$, to find y . Assume $y = Az^n + Bz^{n+r} + Cz^{n+s} \quad \& c.$ then putting Az^n and its Fluxion for y and $\dot{y}$; and you have $enAz^{n-1} + fnAz^{m+n-1} + dAz^{m+n-1} - z^p = 0$; take the Indices $n-1 = p$, then $n = p+1$, and the Indices become $p, p+m$, and the common Difference $= m$, and $r, s, t = m, 2m, 3m, \& c.$ Whence $y = Az^{p+1} + Bz^{p+m+1} + Cz^{p+2m+1}, \& c.$ therefore

$$+ ey$$

$$\begin{aligned}
 + ey &= \overline{p+1.e} Az^p + \overline{p+1+m.e} Bz^{p+m} + \overline{p+1+2m.e} Cz^{p+2m} \&c. \\
 + fz^m y &= \overline{p+1.f} Az^{p+m} + \overline{p+1+m.f} Bz^{p+2m} \&c. \\
 + dyz^{m-1} \dot{z} &= \overline{p+1.d} Az^{p+m} + \overline{p+1+m.d} Bz^{p+2m} \&c. \\
 - z^p \dot{z} &= -z^p
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence } A &= \frac{1}{\overline{p+1.e}}, \quad B = -\frac{d + \overline{p+1.f}}{\overline{p+1+m.e}} A, \\
 &= -\frac{d + \overline{p+1+m.f}}{\overline{p+1+2m.e}} B, \&c. \text{ and putting } \varepsilon = \overline{p+1}
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \delta &= \frac{d}{f} + \overline{p+1}, \text{ then } y = \frac{z^\varepsilon}{\varepsilon e} - \frac{\delta f}{\varepsilon + m.e} Az^{\varepsilon+m} \\
 &- \frac{\delta + m.f}{\varepsilon + 2m.e} Bz^{\varepsilon+2m} - \frac{\delta + 2m.f}{\varepsilon + 3m.e} Cz^{\varepsilon+3m} \&c.
 \end{aligned}$$

Ex. 34.

Let $\dot{x} = \overline{\alpha + \beta z^m}^\mu z^\pi \dot{z}$, to find x by a Series of z
 Suppose $x = y \times \overline{\alpha + \beta z^m}^\mu$, this in Fluxions gives
 $\dot{x} = \dot{y} \times \overline{\alpha + \beta z^m}^\mu + \mu m \beta y z^{m-1} \dot{z} \times \overline{\alpha + \beta z^m}^{\mu-1} =$
 $\overline{\alpha + \beta z^m}^\mu z^\pi \dot{z}$, that is $\alpha \dot{y} + \beta z^m \dot{y} + \mu m \beta y z^{m-1} \dot{z} =$
 $\alpha z^\pi \dot{z} - \beta z^{\pi+m} \dot{z} = 0$. Let $y = Az^n + Bz^{n+r} +$
 $Cz^{n+s} \&c.$ then by Substitution $\alpha n Az^{n-1} +$
 $n \beta Az^{n+m-1} + \mu m \beta A z^{n+m-1} - \alpha z^\pi - \beta z^{\pi+m} = 0$
 And putting the Indices $n-1 = \pi$, then $n = \pi+1$,
 and these Indices are $\pi, \pi+m$, and the common
 Difference m , whence $y = Az^{\pi+1} + Bz^{\pi+1+m} +$
 $Cz^{\pi+1+2m} \&c.$

$$\begin{aligned}
 \text{And } \alpha y &= \overline{\pi+1.\alpha} Az^\pi + \overline{\pi+1+m.\alpha} Bz^{\pi+m} + \overline{\pi+1+2m.\alpha} Cz^{\pi+2m} \&c. \\
 + \beta z^m y &= \overline{\pi+1.\beta} A + \overline{\pi+1+m.\beta} B \&c. \\
 + \mu m \beta y z^{m-1} \dot{z} &= \overline{\pi+1.\mu m \beta} A + \overline{\pi+1+m.\mu m \beta} B \&c. \\
 - \alpha z^\pi \dot{z} - \beta z^{\pi+m} \dot{z} &= -\alpha - \beta
 \end{aligned}$$

Hence

$$\text{Hence } A = \frac{1}{\pi+1}, B = -\frac{\mu m \beta}{\pi+1 \times \pi+1+m \times \alpha},$$

$$C = \frac{\mu m + \pi + 1 + m \times \beta^2 \mu m}{\pi+1 \times \pi+1+m \times \pi+1+2m \times \alpha^2} \mathcal{E}c. \text{ whence}$$

$$\text{putting } q = \frac{\beta z^m}{\alpha}, \varepsilon = \pi+1, \delta = \varepsilon + \mu m. P, Q, R,$$

each preceding Term with its Sign, then $x = \overline{\alpha + \beta z^m}^\mu$

$$\times : \frac{z^\varepsilon}{\varepsilon} - \frac{\mu m q}{\varepsilon+m} P - \frac{\delta+m.q}{\varepsilon+2m} Q - \frac{\delta+2m.q}{\varepsilon+3m} R -$$

$$\mathcal{E}c. = \text{Fl: } \overline{\alpha + \beta z^m}^\mu z^\pi \dot{z}$$

Ex. 35.

$$\text{Let } \dot{x} = e + f z^n + g z^{2n} + h z^{3n} \mathcal{E}c. \times \overline{\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n}}^\mu \mathcal{E}c.$$

$$\times z^\pi \dot{z}, \text{ to find } x \text{ by } z. \text{ Suppose } x = y. \overline{\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n}}^\mu \mathcal{E}c.$$

this put into Fluxions, and then the whole divided

$$\text{by } \overline{\alpha + \beta z^n + \gamma z^{2n}}^\mu \mathcal{E}c. \text{ you have } z^\pi \dot{z} \times e + f z^n + g z^{2n} \mathcal{E}c.$$

$$= \dot{y} \times \overline{\alpha + \beta z^n + \gamma z^{2n}}^\mu \mathcal{E}c. + \mu+1. y \times \overline{\beta z^{n-1} + 2n \gamma z^{2n-1}}^\mu \mathcal{E}c.$$

$$\text{Let } y = A z^m + B z^{m+r} + C z^{m+s} \mathcal{E}c. \text{ then } A z^m \text{ and}$$

its Fluxion being substituted for y and $\dot{y}$, for the first

Term of each compound Quantity in the foregoing

Equation, and there will be $-e z^\pi + m \alpha A z^{m-1} +$

$$\mu+1 \times \beta A z^{m+n-1} \mathcal{E}c. = 0; \text{ and making the Indices}$$

π and $m-1$ equal, then $m = \pi+1$, and the other In-

dex is $\pi+1+n$, and the common Difference n , whence

$$y = A z^{\pi+1} + B z^{\pi+1+n} + C z^{\pi+1+2n} \mathcal{E}c. \text{ And put-}$$

$$\text{ting } \pi+1 = p, \text{ then } \dot{y} = p A z^\pi + p+n \times B z^{\pi+n} + p+2n$$

$$\times C z^{\pi+2n} \mathcal{E}c. \text{ put } m = \mu+1, \text{ then}$$

$$\begin{array}{l|l}
 -ez^p - fz^{p+n} \mathcal{E}c. & -ez^p \quad -fz^{p+n} \quad -gz^{p+2n} \mathcal{E} \\
 + ay & + p\alpha A + \overline{p+n} \times \alpha B + \overline{p+2n} \times \alpha C \mathcal{E} \\
 + \beta z^n y & + p\beta A \quad + \overline{p+n} \times \beta B \mathcal{E} \\
 + \gamma z^{2n} y & + p\gamma A \mathcal{E} \\
 \mathcal{E}c. & \\
 + mn\beta y z^{n-1} & + mn\beta A \quad + mn\beta B \mathcal{E} \\
 + 2mn\gamma y z^{2n-1} & + 2mn\gamma A \mathcal{E} \\
 \mathcal{E}c. &
 \end{array}$$

$$\begin{aligned}
 \text{Hence } A &= \frac{e}{p\alpha}, \quad B = \frac{f - \frac{p}{mn} \beta A}{p + n \times \alpha}, \quad C = \frac{g - \frac{p}{2mn} \gamma A - \frac{p+n}{mn} \beta A}{p + 2n \times \alpha} \\
 D &= \frac{h - \frac{p}{3mn} \delta A - \frac{p+n}{2mn} \gamma B - \frac{p+2n}{mn} \beta C}{p + 3n \times \alpha}, \quad E = \frac{k - \frac{p}{4mn} \epsilon A - \frac{p+n}{3mn} \delta B - \frac{p+2n}{2mn} \gamma C - \frac{p+3n}{mn} \beta D}{p + 4n \times \alpha}, \quad \mathcal{E}
 \end{aligned}$$

$$\text{And } x = Az^p + Bz^{p+n} + Cz^{p+2n} + Dz^{p+3n} \mathcal{E}c.$$

$$\frac{a + \beta z^n + \gamma z^{2n} + \delta z^{3n} \mathcal{E}c.}{\mu + 1}$$

IV.

The *Fluent* of an *irrational Fluxion* may sometimes also be found by *assuming an indetermin'd Series* as in the last Rule.

Ex. 36.

Suppose $\dot{z} = v^2 x^2 \dot{x}$, where $\dot{v} = \frac{ax}{y}$, and $y = \sqrt{2ax - xx}$. I take $Av^2 x^3$ for the first Term, and assume as many Terms of the inferior Powers of v and x , or their Products, as I think will be sufficient for which no general Rule can be given. But you need take no more than the first Power of y , because all the Powers above will be express'd by the Powers

of x , which are supposed to be already in the Equation. Thus I assume $z = Av^2x^3 + Bv^2 + Cx^2 + Dx + E$
 $\times vy + Fx^3 + Gx^2 + Hx = \text{Flut. } v^2x^2\dot{x}$. Put this
 Equation into Fluxions making $\dot{x} = 1$, and writing
 every where $\frac{a}{y}$ for $\dot{v}$; then reduce it from Fractions,
 writing $2ax - xx$ for yy where it occurs. Then col-
 lect severally all the homologous Terms (or those of
 the same Powers of all or any of the Quantities x , v ,
 y), thus

$$\begin{aligned} & + 3Av^2x^2y - 1 \quad + 2aAx^3v - 3C \quad + 5aCx^2v - 2D \quad + 3aDxv - E \quad + 2aBv \\ & + 3Fx^3y + aDxy + aEy \} = 0. \\ & + aC \quad + 2G \quad + H \end{aligned}$$

The respective Coefficients then being equated,
 there will be found $A = \frac{1}{3}$, $C = \frac{2}{9}a$, $D = \frac{5}{9}aa$,
 $E = \frac{5}{3}a^3$, $B = -\frac{5}{6}a^3$, $F = -\frac{2}{27}aa$, $G = -\frac{5}{18}a^3$,
 $H = -\frac{5}{3}a^4$. And thence $z = \frac{x^3v^2}{3} - \frac{5}{6}a^3v^2 +$
 $\frac{1}{3}ax^2 + \frac{5}{9}aax + \frac{5}{3}a^3 \times vy - \frac{2}{27}a^2x^3 - \frac{5}{18}a^3x^2 - \frac{5}{3}a^4x$.

Note, If any of the Quantities B, C, D &c. come
 out equal to nothing, still the Series will be true, pro-
 vided they don't destroy the Quantity A. But if A
 vanish by reason of some of the other Quantities being
 nothing; or if they involve some impossible Equations,
 then the Series is not true; and you must try again by
 assuming more Terms of the Powers or Products of
 x , v , y . But in many Cases it cannot be done in finite
 Terms.

V.

In a fluxionary Equation where the variable Quantity
 is very great, and you would express the Fluent by an
 ascending Series: Or in any very much compounded
 fluxionary

fluxionary Quantity whose Fluent is required; take a given Quantity extremely *near* equal to the variable Quantity; then instead of that variable Quantity *substitute* into the *Equation* the *Sum* of this given Quantity and a new variable Quantity, and likewise the Fluxion for the Fluxion; then find the *Fluent* in simple Terms, and this will be a *Part* of the whole *Fluent* required; and the Operation *repeated* as often as necessary will give the whole *Fluent*.

Ex. 37.

Let $\dot{z} = \dot{x}\sqrt{aa+xx}$; suppose r very near equal to x , and put $r + v = x$, and $aa+rr=ss$, then $\dot{x} =$
and $\dot{z} = \dot{v} \times \sqrt{aa+rr+2rv+vv} = \dot{v}\sqrt{ss+2rv+vv}$

$$= \dot{v} \times : s + \frac{2rv+vv}{2s} - \frac{(2rv+vv)^2}{8s^3} + \frac{2rv+vv}{16s^5}$$

$$- \&c. = \dot{v} \times : s + \frac{rv}{s} + \frac{ss-rr}{2s^3} v^2 - \frac{ss-rr}{2s^5} v^4$$

$$+ \&c. = s\dot{v} + \frac{rv\dot{v}}{s} + \frac{aav^2\dot{v}}{2s^3} - \frac{aarv^3\dot{v}}{2s^5} \&c.$$

$$\text{Whence } z = sv + \frac{rv^2}{2s} + \frac{aav^3}{6s^3} - \frac{aarv^4}{8s^5} \&c.$$

Now in this, substitute $-v$ for v , and subtract the Result from the last Equation, and then $z = 2sv - \frac{a^2v^3}{3s^3} - \&c.$ And this is the Part of the Fluent corresponding to the Difference of the Quantities r and $r-v$, or to $2v$, that is, to these two different Values of x . Hence if v be taken extremely small and there be assumed successively for r , the Numbers or Quantities $b, c, d, e, f, \&c.$ in Arithmetic Progression, whose common Difference is $2v$; that is, so that $2v=b-c=c-d=d-e, \&c.$ till f (the last Value of r) be $=v$; then the Sum of all the Parts corresponding to each (collected by the foregoing Series) will be the whole *Fluent* z required. Or

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you will, you may express the Quantities $b, c, d, e, \&c.$ by $r, r', r'', r''', \&c.$

Ex. 38.

Suppose $\dot{z} = \frac{1}{2}\dot{x}\sqrt{\frac{pp + d - \frac{bx}{a}}{2ax - xx}}$; let $r + v = x$.

$$\text{then } \dot{z} = \frac{1}{2}\dot{v}\sqrt{\frac{pp + dd - \frac{2dbr}{a} - \frac{2dbv}{a} + \frac{bb}{aa} \times r + v}{2ar + 2av - rr - 2rv - vv}} =$$

$$(\text{putting } ss = pp + dd - \frac{2dbr}{a} + \frac{bbrr}{aa}, t = \frac{2bbr}{aa} - \frac{2db}{a},$$

$$qq = 2ar - rr, \text{ and } n = 2a - 2r,) \frac{1}{2}\dot{v}\sqrt{\frac{ss + tv + \frac{bb}{aa}vv}{qq + nv - vv}}.$$

Then dividing the Numerator by the Denominator, and putting $e = \frac{tqq - nss}{q^4}, f = \frac{bb}{aaqq} + \frac{ss}{q^4} +$

$$\frac{n^2s^2 - ntq^2}{q^6}, \text{ we have } \dot{z} = \frac{1}{2}\dot{v}\sqrt{\frac{ss}{qq} + ev + fv^2} \&c. =$$

$$\frac{1}{2}\dot{v} \times \left(\frac{s}{q} + \frac{qe}{2s}v + \frac{4qfs^2 - q^3e^2}{8s^3}v^2 \&c. \right) \text{ Whence}$$

$$z = \frac{sv}{2q} + \frac{qe}{8s}v^2 + \frac{4fs^2 - q^2e^2}{48s^3}qv^3 \&c. \text{ In which}$$

substituting $-v$ for v , and subtracting the Result from the Equation, we have $z = \frac{sv}{q} + \frac{4fs^2 - q^2e^2}{24s^3}qv^3 + \&c.$ for that Part of the Fluent belonging to $2v$, or to the Difference of the Quantities $r + v$ and $r - v$.

Where it is difficult to get many Terms of the Series, as in the last Example, v must be taken so much the smaller, and the Operations oftner repeated before we can obtain the whole Fluent: And the working with Numbers

Numbers instead of known Letters, may sometimes be preferable, when the Quantities are very complex.

Besides the *general Rules* before delivered, there are some *particular Rules* which in some Cases will find the Fluent in *finite Terms*. As

VI.

When one of the *variable Quantities* is *wanting* in the Equation: Then *assume* for its Fluxion the Product of the *other Fluxion* and a *new variable Quantity*, which substitute for the other; and you will get an Equation which put into Fluxions, will give the *Value* of the exterminated Fluxion; and then the Fluent will give the *assumed Quantity*; and from thence the *other Quantity* will be had.

Ex. 39.

Let $yy^3\dot{x} = ax^4 + 2ax^2y^2 + ay^4$, where x is wanting.

Assume $\frac{zy}{a} = \dot{x}$, and expunging $\dot{x}$, $aazy = z^4 +$

$2a^2z^2 + a^4$, whence $y = \frac{z^3}{aa} + 2z + \frac{aa}{z}$. In Fluxions

$\dot{y} = \frac{3z^2\dot{z}}{aa} + 2\dot{z} - \frac{aa\dot{z}}{zz}$, therefore $\frac{zy}{a}$ or $\dot{x} = \frac{3z^3\dot{z}}{a^3}$

$+ \frac{2z\dot{z}}{a} - \frac{a\dot{z}}{z}$. Whence the Fluent is $x = \frac{3z^4}{4a^3} +$

$\frac{z^2}{a} - a \times 2.302585 \text{ Log. } z$. therefore y being known,

z will be known (by the Equation $y = \frac{z^3}{aa} + 2z +$

$\frac{aa}{z}$), and consequently x by the last Equation.

VII. Some-

VII.

Sometimes the *Fluent* may be had, by first putting the Equation into *Fluxions*, making some of the Fluxions *invariable*.

Ex. 40.

Let $\frac{ax + y\dot{x}}{\dot{y}} = x + y - \frac{xy}{x}$. Make y constant,

and put the Equation into Fluxions, then $\frac{a + y \times \ddot{x}}{\dot{y}}$

$+ \dot{x} = \dot{x} + \dot{y} + \frac{xy\ddot{x} - \dot{x}^2\dot{y}}{\dot{x}^2}$, = and $\frac{a + y \times \ddot{x}}{\dot{y}} = \frac{xy\ddot{x}}{\dot{x}^2}$,

whence $a + y \times \ddot{x} = xy\ddot{x}$, and $\frac{\dot{x}}{\sqrt{x}} = \frac{\dot{y}}{\sqrt{a + y}}$, and

the *Fluent* is $\sqrt{x} = \sqrt{a + y}$.

VIII.

Sometimes the *Fluent* may be found by *assuming* other *variable* Quantities to make up the *Fluent*, and finding their Values by Help of the *given* Equations and their Fluxions.

Ex. 41.

Suppose $y\dot{x} - x\dot{x} = ay$. To find x .

Divide be $y - x$, and $\dot{x} = \frac{ay}{y - x}$. Assume v ,

and suppose the *Fluent* $x = a \times 2.302585 \text{ Log. } y - x + v$.

then will $\dot{x} = \frac{ay - a\dot{x} + av}{y - x + v}$ (see Prob. 2. Sect. II.)

and by multiplying, $y\dot{x} - x\dot{x} + v\dot{x} = ay - a\dot{x} + av$,
from this subtract the given Equation $y\dot{x} - x\dot{x} = ay$,
and there remains $v\dot{x} = -a\dot{x} + av$, whence if $\dot{v} = 0$,
then

then will $v = -a$; therefore $x = 2.302585a \times \text{Log} \frac{y}{y-x-a}$.

Ex. 42.

Suppose $az = z\dot{x} - x\dot{z}$. Assume $z = a + x +$
 then $\dot{z} = \dot{x} + \dot{v}$, the Values of z and $\dot{z}$ substituted
 in the given Equation give $ax + av = ax + x\dot{x} +$
 $-x\dot{z}$, that is $av = v\dot{x}$, or $\dot{x} = \frac{av}{v} = \frac{acv}{cv}$, c be-
 ing some given Quantity; whence $x = a \times 2.302585$
 $\text{Log} \frac{v}{c}$, therefore $z = a + v + a \times 2.302585 \text{Log} \frac{v}{c}$.
 Or $z = a + x + c \times \text{Number of the Logarithm}$
 $\frac{x}{2.30258a}$.

Ex. 43.

Suppose $\dot{z} = X^n x^m \dot{x}$, where X is the Hyperbol-
 Logarithm of x . Assume $z = \frac{X^n x^{m+1}}{m+1} + s$; th
 put into Fluxions there arises $\dot{z} = X^n x^m \dot{x}$
 $\frac{nX^{n-1} \dot{X} \times x^{m+1}}{m+1} + \dot{s} = X^n x^m \dot{x}$, therefore $\dot{s}$
 $\frac{-nX^{n-1} x^m \dot{x}}{m+1}$ (writing $\frac{\dot{x}}{x}$ for $\dot{X}$); Again assum
 $s = \frac{-nX^{n-1} x^{m+1}}{m+1^2} + t$, this in Fluxions gives
 $= \frac{n \times \overline{n-1} \times X^{n-2} x^m \dot{x}}{m+1^2}$. Then again t
 $\frac{n \times \overline{n-1} \times X^{n-2} x^{m+1}}{m+1^3} + u$. Whence $\dot{u}$
 $\frac{-n \times \overline{n-1} \times \overline{n-2} \times X^{n-3} x^m \dot{x}}{m+1^3}$, and u

$$- \frac{n \times n-1 \times n-2 \times X^{n-3} x^{m+1}}{m+1^4} + w, \text{ \&c.}$$

$$\text{Whence } z = \frac{X^n x^{m+1}}{m+1} - \frac{n X^{n-1} x^{m+1}}{m+1^2} +$$

$$\frac{n \times n-1 \times X^{n-2} x^{m+1}}{m+1^3} - \frac{n \times n-1 \times n-2 \times X^{n-3} x^{m+1}}{m+1^4}$$

+ &c.

IX.

For compounded *fluxional Quantities*, assume an Equation with *indetermin'd Coefficients*, to comprehend the *Fluxion* given; wherefore the *Equation* must be such that when put into Fluxions the *highest Power* may be the same as that of the Fluxion proposed. Then comparing the homologous Terms, the *Coefficients* will be determin'd. And sometimes the *Indices* may be found this Way.

Ex. 44.

To find the Fluent of $\overline{pyy + qay} \sqrt{yy + ay + aa} \times \dot{y}$.

Assume the imminent Equation $\overline{Ay + B} \times \overline{yy + ay + aa}^{\frac{3}{2}}$, in Fluxions

$$\left. \begin{array}{l} Ayy + Aay + Aaa \\ 3Ayy + 3By \\ + \frac{3}{2}Aay + \frac{3}{2}Ba \end{array} \right\} \dot{y} \sqrt{yy + ay + aa} =$$

$$2yy + qay \times \dot{y} \sqrt{yy + ay + aa}.$$

$$\text{Here } 4A = p, Aa + 3B + \frac{3}{2}Aa = qa, Aa^2 + \frac{3}{2}Ba = 0.$$

$$\text{Whence } A = \frac{1}{4}p, B = \frac{-ap}{6}, \text{ and likewise } B = \frac{1}{3}qa$$

$-\frac{5}{24}ap$. Now if the Fluent be possible, these two Values of B must be equal, whence $8q = p$.

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If the Fluent cannot be had this Way, assume the Form of it different. If $Ay^3 + By^2 + Cy + D\sqrt{yy + ay + aa}$, were assumed, then $p = 8q$ as before.

If $\overline{pyy + qa} \sqrt{yy + ay + aa} \times \dot{y}$ be proposed. If $Ay^3 + B\sqrt{yy + ay + aa}^{\frac{3}{2}}$ be assumed, the Fluent can only be found when $16q = -p$.

If $Ay^3 + By^2 + Cy + D\sqrt{yy + ay + aa}$, the Fluent can only be had when $3p = 80q$.

Ex. 45.

To find the Fluent of $\overline{ax + by} \times \dot{x} + \overline{cx + dy} \times \dot{y}$

= 0. Assume for the Fluent $\overline{x + \alpha y}^\pi \times \overline{x + \beta y}^\tau =$ a given Quantity; and taking the Logarithm $\pi L: \overline{x + \alpha y} + \tau L: \overline{x + \beta y} = L: A$. In Fluxion

$$\frac{\pi \dot{x} + \alpha \pi \dot{y}}{\overline{x + \alpha y}} + \frac{\tau \dot{x} + \beta \tau \dot{y}}{\overline{x + \beta y}} = 0; \text{ reduced } \frac{\pi}{\overline{x + \alpha y}} + \frac{\tau}{\overline{x + \beta y}} \times$$

$$+ \beta \tau + \alpha \tau \times y \dot{x} + \alpha \pi + \beta \tau \times x \dot{y} + \alpha \beta \pi + \alpha \beta \tau \times y \dot{y} =$$

and comparing the Terms with those of the given Equation, when reduced; you'll have 4 Equations

$$\pi + \tau = a, \beta \times \pi + \tau = b, \alpha \pi + \beta \tau = c, \alpha \beta \pi + \alpha \beta \tau = d$$

$$\text{Whence putting } m = \sqrt{b^2 + c^2 - 4ad}, \alpha = \frac{b + c - m}{2a}$$

$$\beta = \frac{b + c + m}{2a}, \pi = \frac{ab - ac + am}{2m}, \tau = \frac{-ab + ac + am}{2m}$$

$$\text{Then the Fluent is } \overline{2ax + b + c - m \times y}^{b - c + m}$$

$$\overline{2ax + b + c + m \times y}^{c - b + m} = B \text{ a given Quantity}$$

After the same Manner the Fluent of $\overline{ax^2 + bxy + cyy}$

$$\times \dot{x} + \overline{dx^2 + fxy + gyy} \times \dot{y} = 0, \text{ may be found}$$

assuming the Fluent $\overline{x + \alpha y}^\pi \times \overline{x + \beta y}^\tau \times \overline{x + \gamma y}^\rho = A$ a constant Quantity; and so of others consisting of more Terms.

Ex. 46.

To find the Fluent of $\frac{ax + bx^2}{cx + xx}$;

Let the Fluent be supposed $d \times \text{Hyp. Log. } cx^r + x^{r+1}$; in Fluxions $d \times \frac{rcx^{r-1}\dot{x} + r+1 \times x^r\dot{x}}{cx^r + x^{r+1}}$ or

$$\frac{drc\dot{x} + d \times r+1 \times x^r\dot{x}}{cx + xx} = \frac{ax + bx^2}{cx + xx}; \text{ then } drc$$

$= a$, and $d \times r+1 = b$. Whence $r = \frac{a}{bc-a}$,

and $d = \frac{bc-a}{c}$. Therefore the Fluent required

$$\text{is } \frac{bc-a}{c} \times \text{Hyp. Log. } cx^{\frac{a}{bc-a}} + x^{\frac{bc}{bc-a}}.$$

Here follow some more Examples wherein the Rules before laid down are promiscuously used.

Ex. 47.

To find the Fluent of $\dot{z} F: \dot{z} F: \dot{z} F: \dot{z} \&c. \times F: y\dot{z}$.

Let $A = F: y\dot{z}$

$$B = z\dot{A} = yz\dot{z}$$

$$C = z\dot{B} = z^2\dot{A} = yz^2\dot{z}$$

$$D = z\dot{C} = z^2\dot{B} = z^3\dot{A} = yz^3\dot{z}$$

$\&c.$

And $\alpha = F: A\dot{z}$, $\beta = F: \alpha\dot{z}$, $\gamma = F: \beta\dot{z}$, $\delta = F: \gamma\dot{z}$, $\&c.$

and let $b = F: z\dot{z}$, $c = F: z\dot{\beta}$, $d = F: z\dot{\gamma}$, $\&c.$ then

1. $F: A\dot{z} = zA - B$, as is evident by putting the Equation into Fluxions; for the same Reason $F: \alpha\dot{z} = z\alpha - b$, $F: \beta\dot{z} = z\beta - c$, $F: \gamma\dot{z} = z\gamma - d$, $\&c.$

2. Since $\alpha = zA - B = F: A\dot{z}$, and b or $F: z\dot{\alpha} = F: A\dot{z}\dot{z}$. Suppose the $F: A\dot{z}\dot{z} = \frac{1}{2}z^2\dot{A} + s$, this in

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Fluxions gives $\dot{s} = -\frac{z^2 \dot{A}}{2} = -\frac{1}{2} \dot{C}$, therefore $s = -\frac{1}{2} C$. Whence $b = \frac{1}{2} z^2 A - \frac{1}{2} C$. And $F: \alpha \dot{z} = (z\dot{z} - b) = z^2 A - zB - \frac{1}{2} z^2 A + \frac{1}{2} C = \frac{1}{2} z^2 A - zB + \frac{1}{2} C = \beta$.

3. $F: \beta \dot{z} = z\beta - c$; but c or $F: z\beta = F: \alpha z \dot{z} = F: zA - B \times z\dot{z}$; suppose its Fluent $= \frac{1}{3} z^3 A - \frac{1}{2} z^2 B + s$, this in Fluxions gives $\dot{s} = -\frac{1}{3} z^3 \dot{A} + \frac{1}{2} z^2 \dot{B} = \frac{1}{6} \dot{D}$. Whence $s = \frac{1}{6} D$; and $c = \frac{1}{3} z^3 A - \frac{1}{2} z^2 B + \frac{1}{6} D$, and therefore $F: \beta \dot{z} = \frac{z^3 A - 3z^2 B + 3zC - D}{6} = \gamma$.

4. By the same Method $F: \gamma \dot{z} = \frac{z^4 A - 4z^3 B + 6z^2 C - 4zD + E}{24}$. And so of others.

Whence

$$1 \quad F: y \dot{z} = A.$$

$$2 \quad F: \dot{z} F: y \dot{z} = zA - B = \alpha = F: A \dot{z}.$$

$$3 \quad F: \dot{z} F: \dot{z} F: y \dot{z} = \frac{z^2 A - 2zB + C}{1 \times 2} = F: \alpha \dot{z} = \beta.$$

$$4 \quad F: \dot{z} F: \dot{z} F: \dot{z} F: y \dot{z} = \frac{z^3 A - 3z^2 B + 3zC - D}{1 \times 2 \times 3} = F: \beta \dot{z} = \gamma.$$

$$5 \quad F: \dot{z} F: \dot{z} F: \dot{z} F: \dot{z} F: y \dot{z} = \frac{z^4 A - 4z^3 B + 6z^2 C - 4zD + E}{1 \times 2 \times 3 \times 4} = F: \gamma \dot{z} = \delta.$$

$\&c.$

After a like Manner the $F: z^{n-1} \times F: y \dot{z} =$

$$\frac{z^n}{n} \times F: y \dot{z} - \frac{F: y z^n \dot{z}}{n}.$$

Ex. 48.

Supposing $A = F: y \dot{z}$, $\alpha = F: \dot{z} F: y \dot{z}$, $\beta = F: \dot{z} F: \dot{z} F: y \dot{z}$, $\gamma = F: \dot{z} F: \dot{z} F: \dot{z} F: y \dot{z}$, $\&c.$ as before. To find the Fluent of $z^n y$.

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Put $F: z^n \dot{y} = z^n y + s$; in Fluxions $z^n \dot{y} = z^n \dot{y} + n z^{n-1} \dot{z} + \dot{s}$, therefore $\dot{s} = -n z^{n-1} y \dot{z} = -n z^{n-1} A$, and $s = -n z^{n-1} A + t$, in Fluxions $\dot{t} = n \times n-1 \times A z^{n-2} \dot{z} = n \times n-1 \times z^{n-2} \alpha$, and $t = n \times n-1 \times z^{n-2} \alpha + v$; in Fluxions $\dot{v} = -n \times n-1 \times n-2 \times z^{n-3} \alpha \dot{z} = -n \times n-1 \times n-2 \times z^{n-3} \beta$, and $v = -n \times n-1 \times n-2 \times z^{n-3} \beta + w$, and $w = n \times n-1 \times n-2 \times n-3 \times z^{n-4} \gamma$, &c. Whence

$$F: z^n \dot{y} = z^n y - n z^{n-1} A + n \times n-1 \times z^{n-2} \alpha - n \times n-1 \times n-2 \times z^{n-3} \beta + n \times n-1 \times n-2 \times n-3 \times z^{n-4} \gamma - \text{&c.}$$

Ex. 49.

Let $3x^3 \dot{y} - 2axy \dot{y} = 3yx^2 \dot{x} - ay^2 \dot{x}$. Put $y = zx$, and $\dot{y} = z \dot{x} + x \dot{z}$, and expunging $y, \dot{y}$; $3x^2 \dot{z} - 2axz \dot{z} = -az^2 \dot{x}$. Again put $x = zv$, and $\dot{x} = z \dot{v} + v \dot{z}$, which substituted in the last Equation, it becomes $3v^2 \dot{z} - 3av \dot{z} = az \dot{v}$.

Let $v = \frac{aa}{s}$, and $\dot{v} = \frac{-aas}{ss}$, and the last Equation becomes $3a\dot{z} - 3s\dot{z} = -z\dot{s}$.

Let $s - a = t$, and $\dot{s} = \dot{t}$, and you'll get the Equation $3t\dot{z} = z\dot{t}$, or $3t\dot{z} - z\dot{t} = 0$, or $\frac{3\dot{z}}{z} - \frac{\dot{t}}{t} = 0$, and the Fluent $z^3 t^{-1}$ or $\frac{z^3}{t} = \frac{1}{a}$. Whence by Restitution, $y^3 + x^3 = axy$.

Ex. 50.

To find the Fluent of $az\dot{x} + \overline{b+rx} \times \dot{z} = \overline{e+fx} \times \dot{x}$; put $y = b+rx$, then the Equation is transform'd into

into $\frac{a}{r}zy + y\dot{z} = \overline{e+fx} \times \dot{x}$; multiply by $y^{\frac{a}{r}-1} = \frac{1}{\overline{b+rx}^{\frac{a}{r}-1}}$, then will $\frac{a}{r}zy^{\frac{a}{r}-1} + y^{\frac{a}{r}}\dot{z} = \overline{e+fx} \times \dot{x} \times \overline{b+rx}^{\frac{a}{r}-1}$. And the Fluent is $zy^{\frac{a}{r}} = \frac{\overline{b+rx}^{\frac{a}{r}}}{a} \times \overline{e+fx} - \frac{f}{aa+ra} \times \overline{b+rx}^{\frac{a}{r}+1}$; divide by $y^{\frac{a}{r}} = \overline{b+rx}^{\frac{a}{r}}$, and you'll have $z = \frac{e+fx}{a} - \frac{f \times \overline{b+rx}}{aa+ar}$.

Ex. 51.

To find the Fluent of $\frac{ax}{x} + \frac{by}{y} - \frac{cz}{z} = 0$

Multiply the Equation by $\frac{x^a y^b}{z^c}$, and we have $\frac{ax^{a-1}y^b\dot{x}}{z^c} + \frac{by^{b-1}x^a\dot{y}}{z^c} - \frac{cx^a y^b \dot{z}}{z^{c+1}}$; and the Fluent is $\frac{x^a y^b}{z^c} = A$ some given Quantity.

Ex. 52.

To find the Fluent of $\frac{\dot{p}}{p} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} - \frac{\dot{z}}{z}$.

The Fluent is $\text{Log. } p = \text{Log. } x + \text{Log. } y - \text{Log. } z$
 $\text{Log. } z = \text{Log. } \frac{xy}{z}$; and therefore $p = \frac{xy}{z}$.

Ex. 53.

 To find the Fluent of $\frac{\alpha + \beta z^n}{\mu + 1} z^{\pi} \dot{z} = \dot{F}$.

 Let $F = \frac{\alpha + \beta z^n}{\mu + 1 \times n\beta} z^{\pi+1-n} + s$; this in Fluxions

 gives $\dot{s} = - \frac{\pi + 1 - n \times \alpha + \beta z^n}{\mu + 1 \times n\beta} z^{\pi-n} \dot{z}$, suppose

 $\dot{s} = - \frac{\pi + 1 - n \times \alpha + \beta z^n}{\mu + 1 \times \mu + 2 \times n^2 \beta^2} z^{\pi+1-2n} + t$, in Fluxions

 $\dot{t} = \frac{\pi + 1 - n \times \pi + 1 - 2n \times \alpha + \beta z^n}{\mu + 1 \times \mu + 2 \times n^2 \beta^2} z^{\pi-2n} \dot{z}$, suppose

 $\dot{t} = \frac{\pi + 1 - n \times \pi + 1 - 2n \times \alpha + \beta z^n}{\mu + 1 \times \mu + 2 \times \mu + 3 \times n^3 \beta^3} z^{\pi+1-3n} +$
 v , &c. Whence $F = \frac{\alpha + \beta z^n}{\mu + 1 \times n\beta} z^{\pi+1-n} + s + t + v$, &c.

 That is Fl: $\frac{\alpha + \beta z^n}{\mu + 1} z^{\pi} \dot{z} = \frac{\alpha + \beta z^n}{\mu + 1 \times n\beta} z^{\pi+1-n} -$
 $\frac{\pi + 1 - n \times \alpha + \beta z^n}{\mu + 2 \times n\beta z^n} A + \frac{\pi + 1 - 2n \times \alpha + \beta z^n}{\mu + 3 \times n\beta z^n} B -$
 $\frac{\pi + 1 - 3n \times \alpha + \beta z^n}{\mu + 3 \times n\beta z^n} C + \text{Ec.}$

Otherwise.

 Suppose $F = \frac{\alpha + \beta z^n}{\pi + 1} z^{\pi+1} + s$, then $\dot{s} = -$
 $\frac{n\beta \times \alpha + \beta z^n}{\pi + 1} z^{\pi+n} \dot{z}$;

Let

$$+ 0 + \frac{\alpha}{\mu} C z^{2n} + \frac{\alpha}{\mu} D z^{3n} \quad \text{Ec. And equating}$$

$$+ \beta z^n + 0 + \frac{\beta}{\mu} C$$

the Coefficients of the several Powers, then $\alpha = \alpha$,

$$\beta = \beta, \gamma = \frac{\alpha}{\mu} C, \delta = \frac{\alpha}{\mu} D + \frac{\beta}{\mu} C, \text{ Ec. Whence}$$

$$C = \frac{\mu\gamma}{\alpha}, D = \frac{\mu\delta}{\alpha} - \frac{\beta\mu\gamma}{\alpha\alpha}, E = \frac{\mu\epsilon}{\alpha} - \frac{1-\mu}{2\mu} CC -$$

$$\frac{1}{2}D, \text{ Ec. Therefore Fl: } \overline{\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n}} \text{ Ec.}^\mu \times$$

$$z^\pi \dot{z} = F: \overline{\alpha + \beta z^n}^\mu z^\pi \dot{z} \times \text{ into } 1 + \frac{\mu\gamma}{\alpha} z^{2n} +$$

$$\frac{\mu\delta - \beta\gamma}{\alpha\alpha} \mu z^{3n} + E z^{4n} \text{ Ec.}$$

Therefore if the Fluent of $\overline{\alpha + \beta z^n}^\mu z^\pi \dot{z}$ be known, the Fluent of this compound Fluxion will be known, by Form 11th of the Table, and this is useful when the Quantities γz^{2n} , δz^{3n} , Ec. are very small.

Likewise the whole Fluent, when $\alpha + \beta z^n$ becomes 0, will be found from this Rule, by Form 17th of the Table.

Ex. 55.

To find the Fluent of $z^n x^m y^m \dot{y}$.

Let $x^m y^m \dot{y} = \dot{A}$, $A \dot{z} = \dot{B}$, $B \dot{z} = \dot{C}$, Ec. then the Fluxion given is $z^n \dot{A}$; then by Rule VIII. put $F: z^n \dot{A} = z^n A + s$; in Fluxions $z^n \dot{A} = z^n \dot{A} + n A z^{n-1} \dot{z} + \dot{s}$, or $\dot{s} = -n z^{n-1} A \dot{z} = -n z^{n-1} \dot{B}$; assume $s = -n z^{n-1} B + t$, then $\dot{t} = n \times \overline{n-1} \times z^{n-2} B \dot{z} = n \times \overline{n-1} z^{n-2} \dot{C}$: Let $t = n \times \overline{n-1} \times z^{n-2} C + u$; then $\dot{u} = -n \times \overline{n-1} \times \overline{n-2} \times z^{n-3} C \dot{z} = -n \times \overline{n-1} \times \overline{n-2} \times z^{n-3} \dot{D}$. Then assume $u = -n \times \overline{n-1} \times \overline{n-2} \times z^{n-3} D + w$, Ec. Then at last $F: z^n x^m y^m \dot{y}$ or $F: z^n \dot{A} = z^n A - n z^{n-1} B +$

L

$n \times$

$$n \times \overline{n-1} \times z^{n-2} C - n \times \overline{n-1} \times \overline{n-2} \times z^{n-3} D + n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times z^{n-4} E - \mathcal{E}c.$$

Then by the very same Rule that these are had, you will get the Fluents of $\dot{A}$ or $x^r y^m \dot{y}$, and $\dot{B}$ or $A \dot{z}$, of $\dot{C}$ or $B \dot{z}$, &c. in any particular Case; expunging by Degrees $\dot{x}$ $\dot{y}$ or $\dot{z}$ as there is Occasion, by Help of the given Equations. The above Series is the same as is found, Example 48.

Suppose the given Fluxion is $zx^2y\dot{y}$, and $rr - yy = \dot{y}\dot{y}$, and $-xx = \dot{y}\dot{y}$, also $x\dot{z} = -r\dot{y}$ and $y\dot{z} = r\dot{x}$; r, y, x being the Radius, Sine and Cosine in the Circle. Then we shall have $F: zx^2y\dot{y} = zA - B$, where $A = F: x^2y\dot{y}$, and $B = F: A\dot{z}$; here $A = F: \overline{rr - yy} \times y\dot{y} = \frac{r^2y^4}{4} - \frac{y^6}{6}$, and $B = F: \frac{1}{4}rr - \frac{1}{6}yy \times y^4\dot{z} = \frac{1}{4}rr - \frac{1}{6}yy$

$\times y^3 \times r\dot{x} = \frac{1}{4}rr - \frac{1}{6}yy \times \frac{-ry^4\dot{y}}{\sqrt{rr - yy}}$, whose Fluent is found from the Table.

And thus I have explain'd, as clearly as I could the most *general Methods* of solving this *difficult Problem*. The Reader perhaps may think me too prolix on this Head: But it must be consider'd that this is a Problem of the *greatest Extent* and *Use* in the whole Practice of Fluxions; nay it contains almost the whole Science, and we cannot be too particular in treating on that which is the *Foundation* of the greatest Part of the *Practice*. But after all, we shall find it exceeding difficult in many Cases to find the Fluents of Quantities by any Methods hitherto known. And it is much to be wish'd that we had some easier Methods of finding Fluents, especially of compound fluxionary Quantities, without the tedious Labour of reducing them to infinite Series, which in many Cases converge so slow, and are so much compounded as to be in a manner useless. The following Proposition is design'd to remedy this Difficulty in some particular Cases.

P R O P

PROP. XI.

To find the *Fluent* of a given *Fluxion* by the *Table*.

1. The following Table comprehends all Sorts of *Fluxions* and their correspondent *Fluents*; not only such as can be exactly had in *finite Terms*, and those depending on the *Quadrature* of the *Conic Sections*, but also those that can only be had by *infinite Series*.

2. These Forms are all number'd in the first Column; the second Column contains the *Fluxions*, and the third gives the *Fluent* thereof. The 21st and all the following Forms relate to the *Transmutation* of *Fluxions*; here the *Fluxions* in the second and third Columns are equal, the second being transform'd into the third, and the *Fluxion* in the third Column always belongs to some of the foregoing Forms.

3. Here z , v , y express *variable Quantities*, and all the rest are *given ones*, which may represent any *Quantities* whatsoever affirmative or negative. But in the 6th, 8th, and 10th Forms, the Nature of them requires *negative Quantities*, and therefore they are written *negative*.

4. In the second Column are set down all the *necessary Conditions* relating to the *Signs*, *Indices*, &c. in each Form: likewise in what Cases the *Form* will fail, and when the *Series* will *terminate*. But the *Fluents* in Form 11, 12, 13, and 14 are design'd always to *terminate*, and are derived from the foregoing ten Forms, which therefore may be called *original Forms*.

5. In these original Forms, though the *first* of the *Fluents* there given may in most Cases be sufficient, yet there are several *Varieties* both of *numerical* and *geometrical Fluents*; so that the most simple and elegant may always be chosen to suit any particular Case. It is sufficient to premise this concerning the Nature of these Forms.

A T A B L E

Of FLUXIONS and FLUENTS.

$$L = 2.302585092994045684 \text{ \&c.}$$

$$N = .017453292519943295 \text{ \&c.}$$

Forms.	Fluxions.	Fluents.
I	$z^{n-1}\dot{z}$ <i>This Form fails when $n = 0$.</i>	$\frac{z^n}{n}$
2	$z^{-1}\dot{z} = \dot{\phi}$	$\phi = L \times \text{Log: } z. \text{ Or } L \times \text{Log: } Az$ Or $\phi = \frac{\text{Area}}{RR}$ of the Hyperbola between the Assymptotes, whose inscribed Parallelogram is any Space RR, and Abscissa (taken in the Assymptote) Rz or z.
3	$\frac{\dot{z}}{a + \beta z^n} z^{n-1}\dot{z}$ <i>This Form fails when $\mu = -1$.</i>	$\frac{(a + \beta z^n)^{\mu+1}}{\mu+1 \times n\beta}$
4	$\frac{z^{n-1}\dot{z}}{a + \beta z^n} = \dot{\phi}$	$\phi = \frac{L}{n\beta} \times \text{Log. of } a + \beta z^n$ $\phi = \frac{\text{Area}}{n\beta RR}$ of an Hyperbola between the Assymptotes, whose inscribed Parallelogram is RR, Abscissa $\frac{a}{\beta} + z^n$.

Fluxions.

Fluents.

$$\frac{z^{\frac{1}{2}n-1} \dot{z}}{\alpha + \beta z^n} = \phi$$

Here α and β
are affirmative.

$$\phi = \frac{2N}{n\sqrt{\alpha\beta}} \times \text{Degrees in that Arch of a Circle whose Radius is 1, and natural Tangent } \sqrt{\frac{\beta z^n}{\alpha}}.$$

$$\phi = \frac{\text{Arch}}{\frac{1}{2}nR\sqrt{\alpha\beta}} \text{ of a Circle whose Radius}$$

is any Line R, and Tangent $R\sqrt{\frac{\beta z^n}{\alpha}}$.

$$\phi = \frac{4 \text{ Sectors}}{nRR\sqrt{\alpha\beta}} \text{ of the Circle whose Radius is R, and Tangent } R\sqrt{\frac{\beta z^n}{\alpha}}.$$

$$\frac{z^{\frac{1}{2}n-1} \dot{z}}{\alpha - \beta z^n} = \phi$$

Here α is affirmative, and $-\beta$
negative.

$$\phi = \frac{L}{n\sqrt{\alpha\beta}} \times \text{vulg. Log. of } \frac{\sqrt{\alpha} + \sqrt{\beta z^n}}{\sqrt{\alpha} - \sqrt{\beta z^n}}.$$

$$\phi = \frac{L}{n\sqrt{\alpha\beta}} \times \text{Log. of } \frac{\sqrt{\beta z^n} + \sqrt{\alpha}}{\sqrt{\beta z^n} - \sqrt{\alpha}}.$$

$$\phi = \frac{L}{n\sqrt{\alpha\beta}} \times \text{Log. of } \frac{\alpha + \beta z^n + 2\sqrt{\alpha\beta z^n}}{\alpha - \beta z^n}.$$

$$\phi = \frac{L}{n\sqrt{\alpha\beta}} \times \text{Log. of } \frac{\alpha - \beta z^n}{\alpha + \beta z^n - 2\sqrt{\alpha\beta z^n}}.$$

$$\phi = \frac{\text{Area}}{nRR\sqrt{\alpha\beta}} \text{ of the Hyperbola be-}$$

tween the Assymptotes, whose inscribed Parallelogram is RR, and Abscissas (terminating this Area) $\sqrt{\alpha} + \sqrt{\beta z^n}$ and $\sqrt{\alpha} \propto \sqrt{\beta z^n}$.

$$\phi = \frac{4 \text{ Sectors}}{nRR\sqrt{\alpha\beta}} \text{ of a right angled Hyperbola, whose semitransverse is R, and Tangent at the Vertex } R\sqrt{\frac{\beta z^n}{\alpha}}.$$

Forms.	Fluxions.
7	$\frac{z^{\frac{\delta}{\lambda}n-1} \cdot z}{\alpha + \beta z^n}$ Here α and β are affirmative. $\frac{\delta}{\lambda}$ is an affirmative proper Fraction.

Fluents.

$$\frac{1}{\lambda - \delta} \times \text{into } AL \times \text{Log. } \sqrt{1 - 2sx + xx} + BN \times P \\ na^{\frac{1}{\lambda}} \beta^{\lambda} + CL \times \text{Log. } \sqrt{1 - 2tx + xx} + DN \times Q \\ + EL \times \text{Log. } \sqrt{1 - 2ux + xx} + FN \times R \text{ \&c.}$$

$$K = \text{an Arch of } \frac{180 \text{ Deg.}}{\lambda}. \quad x = \frac{\beta z^n}{\alpha} \Big|^\frac{1}{\lambda}.$$

$a, b, c, \text{ \&c.} = \text{Sines } \left. \begin{array}{l} \text{of } 1K, 3K, 5K \text{ \&c. to } 180^\circ. \\ s, t, u, \text{ \&c.} = \text{Cosines} \end{array} \right\} \text{Radius} = 1.$

$P, Q, R, \text{ \&c.} = \text{Degrees of Arches whose Sines are}$

$$\frac{ax}{\sqrt{1 - 2sx + xx}}, \quad \frac{bx}{\sqrt{1 - 2tx + xx}}, \quad \frac{cx}{\sqrt{1 - 2ux + xx}}, \text{ \&c.}$$

C A S E S.

1. If $\frac{\delta}{\lambda} = \frac{1}{3}$, then $A = (-2s) - 1. B = 2a. C = 1. D = 0 = E = F \text{ \&c.}$

2. If $\frac{\delta}{\lambda} = \frac{2}{3}$, $A = (2s) 1. B = 2a. C = -1. D, E, F \text{ \&c.} = 0.$

3. If $\frac{\delta}{\lambda} = \frac{1}{4}$, $A = -2s. B = 2a. C = -2t. D = 2b. \left. \begin{array}{l} \\ \\ \end{array} \right\} E, F \text{ \&c.} = 0.$

4. If $\frac{\delta}{\lambda} = \frac{3}{4}$, $A = 2s. B = 2a. C = 2t. D = 2b. \left. \begin{array}{l} \\ \\ \end{array} \right\} E, F \text{ \&c.} = 0.$

5. If $\frac{\delta}{\lambda} = \frac{1}{5}$, $A = -2s. B = 2a. C = -2t. D = 2b. \\ E = 1. F \text{ \&c.} = 0.$

6. If $\frac{\delta}{\lambda} = \frac{2}{5}$, $A = 2t. B = 2b. C = 2s. D = -2a. \\ E = -1. F \text{ \&c.} = 0.$

7. If $\frac{\delta}{\lambda} = \frac{1}{6}$, $A = -2s. B = 2a. C = 0. D = 2b. \\ E = -2u. F = 2c. \text{ \&c.} = 0.$

\&c.

<i>Forms.</i>	<i>Fluxions.</i>
8	$\frac{z^{\frac{\delta}{\lambda}n-1} \cdot z}{\alpha - \beta z^n}$ <i>Here α is affirmative, and $-\beta$ negative.</i> $\frac{\delta}{\lambda}$ is an affirmative proper Fraction.

Fluents.

$$\frac{1}{\frac{\delta}{\lambda} \frac{\lambda - \delta}{\lambda}} \times \text{into } AL \times \text{Log. } \overline{1 - x} + BL \times \\ n\beta^{\frac{\delta}{\lambda}} a^{\frac{\lambda - \delta}{\lambda}} \text{Log. } \sqrt{1 - 2sx + xx} + CN \times P + \\ DL \times \text{Log. } \sqrt{1 - 2tx + xx} + EN \times Q \\ + FL \times \text{Log. } \sqrt{1 - 2ux + xx} + GN \times R \text{ \&c.}$$

$$K = \text{an Arch of } \frac{180 \text{ Deg.}}{\lambda}. \quad x = \frac{\beta z^n}{a}^{\frac{1}{\lambda}}.$$

$a, b, c, \text{ \&c.} = \text{Sines} \quad \left. \begin{array}{l} \text{of } 2K, 4K, 6K \text{ \&c. to } 180^\circ. \\ s, t, u, \text{ \&c.} = \text{Cofines} \end{array} \right\} \text{Radius} = 1.$

$P, Q, R, \text{ \&c.} = \text{Degrees of Arches whose Sines are}$

$$\frac{ax}{\sqrt{1 - 2sx + xx}}, \quad \frac{bx}{\sqrt{1 - 2tx + xx}}, \quad \frac{cx}{\sqrt{1 - 2ux + xx}}, \text{ \&c.}$$

C A S E S.

1. If $\frac{\delta}{\lambda} = \frac{1}{3}$, then $A = -1, B = -2s, C = 2a.$
2. If $\frac{\delta}{\lambda} = \frac{2}{3}$, $A = -1, B = -2s, C = -2a, D, E \text{ \&c.} = 0.$
3. If $\frac{\delta}{\lambda} = \frac{1}{4}$, $A = -1, B = (-2s =) 0, C = 2a, D = 1.$
4. If $\frac{\delta}{\lambda} = \frac{3}{4}$, $A = -1, B = 0, C = -2a, D = 1, E, F \text{ \&c.} = 0.$
5. If $\frac{\delta}{\lambda} = \frac{1}{5}$, $A = -1, B = -2s, C = 2a, D = -2t, E = 2b.$
6. If $\frac{\delta}{\lambda} = \frac{2}{5}$, $A = -1, B = -2t, C = 2b, D = -2s, E = -2a, F \text{ \&c.} = 0.$
7. If $\frac{\delta}{\lambda} = \frac{1}{6}$, $A = -1, B = -2s, C = 2a, D = -2t, E = 2b, F = 1, G, \text{ \&c.} = 0.$

\&c.

<i>Forms.</i>	<i>Fluxions.</i>
9	$\frac{z^{\frac{1}{2}n-1} \cdot z}{\sqrt{a + \beta z^n}} = \dot{\phi}$ <i>Here β is affirmative.</i>

Fluents.

$$\phi = \frac{2L}{n\sqrt{\beta}} \times \text{Log. of } \sqrt{\beta z^n} + \sqrt{\alpha + \beta z^n}.$$

$$\phi = \frac{L}{n\sqrt{\beta}} \times \text{Log. of } \alpha + 2\beta z^n + 2\sqrt{\alpha + \beta z^n} \times \beta z^n.$$

$$\phi = \frac{-2L}{n\sqrt{\beta}} \times \text{Log. of } \sqrt{\alpha + \beta z^n} \cup \sqrt{\beta z^n}.$$

$$\phi = \frac{-L}{n\sqrt{\beta}} \times \text{Log. of } \alpha + 2\beta z^n - 2\sqrt{\alpha \beta z^n} + \beta^2 z^{2n}.$$

$$\phi = \frac{4 \text{ Sectors}}{nRR\sqrt{\beta}} \text{ of the right angled Hyperbola, whose Semitransverse is } R, \text{ and Ordinate } R\sqrt{\frac{\beta z^n}{\alpha}}.$$

$$\phi = \frac{2}{nRR\sqrt{\beta}} \times \text{Sector of the right Hyperbola, whose Semitransverse is } R, \text{ and Ordinate } \frac{2R}{\alpha} \sqrt{\alpha + \beta z^n} \times \beta z^n.$$

$$\phi = \frac{2}{nRR\sqrt{\beta}} \times \text{Sector of the right Hyperbola, whose Semitransverse is } R, \text{ and Ordinate } \frac{2R}{-\alpha} \sqrt{\alpha + \beta z^n} \times \beta z^n, \text{ when } \alpha \text{ stands for a negative Quantity.}$$

$$\phi = \frac{4 \text{ Sectors}}{nRR\sqrt{\beta}} \text{ of the right Hyperbola, whose Semitransverse is } R, \text{ and Ordinate } \frac{R}{\sqrt{-\alpha}} \times \sqrt{\alpha + \beta z^n}, \text{ when } \alpha \text{ stands for a negative Quantity.}$$

Forms.	Fluxions.
10	$\frac{z^{\frac{1}{2}n-1} \dot{z}}{\sqrt{\alpha - \beta z^n}} = \dot{\phi}$ Here α is affirmative, and $-\beta$ is negative.

Fluents.

$$\phi = \frac{2N}{n\sqrt{\beta}} \times \text{Degrees in the Arch of a Circle} \\ \text{whose Radius is 1, and natural Sine} \\ \sqrt{\frac{\beta z^n}{\alpha}}.$$

$$\phi = \frac{N}{n\sqrt{\beta}} \times \text{Degrees in the Arch whose Radius} \\ \text{is 1, and Sine } \frac{2}{\alpha} \sqrt{\alpha - \beta z^n} \times \beta z^n.$$

$$\phi = \frac{2}{nR\sqrt{\beta}} \times \text{Arch whose Radius is R, and} \\ \text{Sine } R\sqrt{\frac{\beta z^n}{\alpha}};$$

$$\phi = \frac{\text{Arch}}{nR\sqrt{\beta}}, \text{ whose Radius is R, and Sine} \\ \frac{2R}{\alpha} \sqrt{\alpha - \beta z^n} \times \beta z^n, \text{ or versed Sine } \frac{2R}{\alpha} \beta z^n.$$

$$\phi = \frac{4}{nRR\sqrt{\beta}} \times \text{Sector of a Circle, whose Radius} \\ \text{is any Line R, and Sine } R\sqrt{\frac{\beta z^n}{\alpha}}.$$

$$\phi = \frac{2 \text{ Sectors}}{nRR\sqrt{\beta}} \text{ of the Circle whose Radius is R,} \\ \text{and Sine } \frac{2R}{\alpha} \sqrt{\alpha \beta z^n - \beta^2 z^{2n}}.$$

Forms.	Fluxions.
I 1	$\overline{\alpha + \beta z^n}^\mu z^{\pi + \lambda n} z$ Here λ is any affirmative whole Number. N. B. This Form fails when $\frac{\pi + 1}{n} + \mu$ is any negative whole Number from -1 to $-\lambda$ inclusive.
I 2	$\overline{\alpha + \beta z^n}^\mu z^{\pi - \lambda n} z$ Here λ is any affirmative whole Number. N. B. This Form fails when $\frac{\pi + 1}{n}$ is any affirmative whole Number from 1 to λ inclusively.

Fluents.

$$\begin{aligned} & \frac{\alpha^\lambda \times \overline{\epsilon - \eta} \times \overline{\epsilon - 2\eta} \times \overline{\epsilon - 3\eta} \times \mathcal{E}c. \dots \text{till } \overline{\epsilon - \lambda\eta}}{\beta^\lambda \times \overline{\delta - \eta} \times \overline{\delta - 2\eta} \times \overline{\delta - 3\eta} \times \mathcal{E}c. \dots \text{till } \overline{\delta - \lambda\eta}} \phi \\ & + \frac{z^{\epsilon - \eta} \gamma}{\overline{\delta - \eta} \times \beta} - \frac{\overline{\epsilon - \eta} \times \alpha}{\overline{\delta - 2\eta} \times \beta z^\eta} A - \frac{\overline{\epsilon - 2\eta} \times \alpha}{\overline{\delta - 3\eta} \times \beta z^\eta} B \\ & - \frac{\overline{\epsilon - 3\eta} \times \alpha}{\overline{\delta - 4\eta} \times \beta z^\eta} C - \mathcal{E}c. \text{ till } - \frac{\overline{\epsilon + \eta - \lambda\eta} \times \alpha}{\overline{\delta - \lambda\eta} \times \beta z^\eta} F. \end{aligned}$$

ϕ = Fluent of $\overline{\alpha + \beta z^\eta}^\mu z^\pi \dot{z}$.

$\gamma = \overline{\alpha + \beta z^\eta}^{\mu + 1}$.

$\epsilon = \pi + 1 + \lambda\eta$.

$\delta = \epsilon + \eta + \mu\eta$.

A, B, C $\mathcal{E}c.$ each preceding Term with its Sign.

$$\begin{aligned} & \frac{\beta^\lambda \times \overline{\delta + \eta} \times \overline{\delta + 2\eta} \times \overline{\delta + 3\eta} \times \mathcal{E}c. \dots \text{till } \overline{\delta + \lambda\eta}}{\alpha^\lambda \times \overline{\epsilon + \eta} \times \overline{\epsilon + 2\eta} \times \overline{\epsilon + 3\eta} \times \mathcal{E}c. \dots \text{till } \overline{\epsilon + \lambda\eta}} \phi \\ & + \frac{z^{\epsilon + \eta} \gamma}{\overline{\epsilon + \eta} \times \alpha} - \frac{\overline{\delta + \eta} \times \beta z^\eta}{\overline{\epsilon + 2\eta} \times \alpha} A - \frac{\overline{\delta + 2\eta} \times \beta z^\eta}{\overline{\epsilon + 3\eta} \times \alpha} B \\ & - \frac{\overline{\delta + 3\eta} \times \beta z^\eta}{\overline{\epsilon + 4\eta} \times \alpha} C - \mathcal{E}c. \text{ till } - \frac{\overline{\delta - \eta + \lambda\eta} \times \beta z^\eta}{\overline{\epsilon + \lambda\eta} \times \alpha} F. \end{aligned}$$

ϕ = Fluent of $\overline{\alpha + \beta z^\eta}^\mu z^\pi \dot{z}$.

$\gamma = \overline{\alpha + \beta z^\eta}^{\mu + 1}$.

$\epsilon = \pi + 1 - \lambda\eta - \eta$.

$\delta = \epsilon + \mu\eta + \eta$.

A, B, C $\mathcal{E}c.$ each preceding Term with its Sign.

Forms.	Fluxions.
I 3	$\overline{a + \beta z^\eta}^{\mu + \tau} z^\pi \dot{z}$ <i>Here τ is any affirmative whole Number.</i> <i>N. B. This Form fails when $\frac{\pi + 1}{\eta} + \mu$ is any negative whole Number from -1 to $-\tau$ inclusively.</i>
I 4	$\overline{a + \beta z^\eta}^{\mu - \tau} z^\pi \dot{z}$ <i>Here τ is any affirmative whole Number.</i> <i>N. B. This Form will fail when $\mu + 1$ is any affirmative whole Number from 1 to τ inclusive.</i>

Fluents.

$$\begin{aligned}
 & \frac{\eta^{\tau} \alpha^{\tau} \times \overline{\mu+1} \times \overline{\mu+2} \times \overline{\mu+3} \times \mathcal{E}c. \dots \text{till } \overline{\mu+\tau}}{\overline{\delta+\eta} \times \overline{\delta+2\eta} \times \overline{\delta+3\eta} \times \mathcal{E}c. \dots \text{till } \overline{\delta+\tau\eta}} \phi \\
 & + \frac{\overline{z^{\tau+1}} \gamma^{\mu+\tau}}{\overline{\delta+\tau\eta}} + \frac{\overline{\mu+\tau} \times \eta\alpha}{\overline{\delta+\tau\eta-\eta} \times \gamma} A + \frac{\overline{\mu+\tau-1} \times \eta\alpha}{\overline{\delta+\tau\eta-2\eta} \times \gamma} B \\
 & + \frac{\overline{\mu+\tau-2} \times \eta\alpha}{\overline{\delta+\tau\eta-3\eta} \times \gamma} C + \mathcal{E}c. \text{ till } \frac{\overline{\mu+2} \times \eta\alpha}{\overline{\delta+\eta} \times \gamma} F.
 \end{aligned}$$

ϕ = Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu} z^{\tau} \dot{z}$.

$$\gamma = \alpha + \beta z^{\eta}.$$

$$\delta = \pi + 1 + \mu\eta.$$

A, B, C &c. each preceding Term with its Sign.

$$\begin{aligned}
 & \frac{\delta \times \overline{\delta-\eta} \times \overline{\delta-2\eta} \times \overline{\delta-3\eta} \times \mathcal{E}c. \dots \text{till } \overline{\delta+\eta-\tau\eta}}{\eta^{\tau} \alpha^{\tau} \times \overline{\mu-1} \times \overline{\mu-2} \times \overline{\mu-3} \times \mathcal{E}c. \dots \text{till } \overline{\mu+1-\tau}} \phi \\
 & - \frac{\overline{z^{\tau+1}} \gamma^{\mu-\tau+1}}{\overline{\mu-\tau+1} \times \eta\alpha} + \frac{\overline{\delta-\tau\eta+\eta} \times \gamma}{\overline{\mu-\tau+2} \times \eta\alpha} A + \frac{\overline{\delta-\tau\eta+2\eta} \times \gamma}{\overline{\mu-\tau+3} \times \eta\alpha} B \\
 & + \frac{\overline{\delta-\tau\eta+3\eta} \times \gamma}{\overline{\mu-\tau+4} \times \eta\alpha} C + \mathcal{E}c. \text{ till } \frac{\overline{\delta-\eta} \times \gamma}{\mu\eta\alpha} F.
 \end{aligned}$$

ϕ = Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu} z^{\tau} \dot{z}$.

$$\gamma = \alpha + \beta z^{\eta}.$$

$$\delta = \pi + 1 + \mu\eta.$$

A, B, C &c. each preceding Term with its Sign.

Forms.	Fluxions.
15	$\overline{\alpha + \beta z^n}^\mu z^\pi \dot{z}$ <i>This Series will terminate when $\frac{\pi+1}{n} + \mu$ is any negative whole Number.</i> N. B. <i>This Form fails when $\frac{\pi+1}{n}$ is any negative whole Number whatsoever.</i>
16	$\overline{\alpha + \beta z^n}^\mu z^\pi \dot{z}$ <i>This Series will terminate when $\frac{\mu}{v}$ is any affirmative whole Number.</i>
17	$\overline{\alpha + \beta z^n}^\mu z^\pi \dot{z} \times \text{into}$ $e + fz^n + gz^{2n} + hz^{3n} \&c.$ <i>Here μ and π must each be greater than —</i> <i>Also z and $\overline{\alpha + \beta z^n}^{\mu+1}$ must be one of them = 0 at the beginning of the Fluent the other = 0 at the End of it.</i> <i>This Series will terminate when $\frac{\pi+1}{n}$ is any negative whole Number whatever.</i>

Fluents.

$$\frac{z^{\varepsilon}}{\varepsilon \alpha} - \frac{\delta + \eta}{\varepsilon + \eta} Aq - \frac{\delta + 2\eta}{\varepsilon + 2\eta} Bq - \frac{\delta + 3\eta}{\varepsilon + 3\eta} Cq \\ - \frac{\delta + 4\eta}{\varepsilon + 4\eta} Dq - \& c. \times \text{into } \overline{\alpha + \beta z^{\eta}}^{\mu+1}.$$

$$\varepsilon = \pi + 1.$$

$$\delta = \varepsilon + \mu \eta.$$

$$q = \frac{\beta z^{\eta}}{\alpha}.$$

A, B, C &c. each preceding Term with its Sign.

$$\frac{\frac{\mu}{\alpha} z^{\pi+1}}{\pi+1} + \frac{\frac{\mu}{\nu} Aq}{\pi+1+\eta} + \frac{\frac{\mu-\nu}{2\nu} Bq}{\pi+1+2\eta} + \frac{\frac{\mu-2\nu}{3\nu} Cq}{\pi+1+3\eta}$$

$$+ \frac{\frac{\mu-3\nu}{4\nu} Dq}{\pi+1+4\eta} + \& c.$$

$$q = \frac{\beta z^{\eta}}{\alpha}.$$

A, B, C &c. are the Numerators of each preceding Term with its Sign.

$$\phi \times e - \frac{\varepsilon \alpha}{\delta \beta} \times f - \frac{\overline{\varepsilon + \eta} \times \alpha}{\delta + \eta \times \beta} Ag - \frac{\overline{\varepsilon + 2\eta} \times \alpha}{\delta + 2\eta \times \beta} Bb$$

$$- \frac{\overline{\varepsilon + 3\eta} \times \alpha}{\delta + 3\eta \times \beta} Ck, \& c.$$

$$\varepsilon = \pi + 1.$$

$$\delta = \pi + 1 + 1 + \mu \times \eta.$$

$$\phi = \text{whole Fluent of } \overline{\alpha + \beta z^{\eta}}^{\mu} z^{\pi} \dot{z}.$$

A, B, C &c. each preceding Term with its Sign, but without the Quantities $f, g, b, \&c.$

When any of the Quantities $e, f, g, \&c.$ are wanted the respective Terms will vanish.

Forms.	Fluxions.
18	$\frac{e + fz^n + gz^{2n} + hz^{3n} \&c \times \text{into}}{a + \beta z^n + \gamma z^{2n} + \delta z^{3n} \&c} z^\pi z,$
	N. B. This Form will fail when $\frac{\pi+1}{n}$ is any negative whole Number whatever, or when any Denominator is 0.
19	$\frac{mzy + nyz \times z^{n-1} y^{m-1}}{\text{or } \frac{my}{y} + \frac{nz}{z} \times z^n y^m}.$
20	$rzyv + mzyv + nyvz \times \text{into } z^{n-1} y^{m-1} v^{r-1}.$

Fluents.

$$+ \frac{e}{p\alpha}$$

$$+ \frac{f - \frac{p}{m\eta} \} \beta A}{p + \eta \times \alpha} z^\eta$$

$$+ \frac{g - \frac{p}{2m\eta} \} \gamma A - \frac{p + \eta}{m\eta} \} \beta B}{p + 2\eta \times \alpha} z^{2\eta}$$

$$+ \frac{h - \frac{p}{3m\eta} \} \delta A - \frac{p + \eta}{2m\eta} \} \gamma B - \frac{p + 2\eta}{m\eta} \} \beta C}{p + 3\eta \times \alpha} z^{3\eta}$$

$$+ \frac{i - \frac{p}{4m\eta} \} \varepsilon A - \frac{p + \eta}{3m\eta} \} \delta B - \frac{p + 2\eta}{2m\eta} \} \gamma C - \frac{p + 3\eta}{m\eta} \} \beta D}{p + 4\eta \times \alpha} z^{4\eta}$$

$$+ \mathcal{E}c. \times \text{into } z^p \times \frac{\alpha + \beta z^\eta + \gamma z^{2\eta} + \delta z^{3\eta}}{\mathcal{E}c.}^{\mu+1}$$

$$p = \pi + 1.$$

$$m = \mu + 1.$$

A, B, C &c. the Coefficients of the 1st, 2d, 3d &c. Terms respectively.

$$z^n y^m.$$

$$z^n y^m v^r.$$

Forms.	Given Fluxions.
21	$\overline{e + fz^\theta}^m z^{\lambda\theta-1} \dot{z}$ λ is any whole Number whatever.
22	$\frac{z^{\lambda\theta-1} \dot{z}}{\overline{e + fz^\theta} \times \overline{g + hz^\theta}}$
23	$\overline{e + fz^\theta}^m \times \overline{g + hz^\theta}^r z^{\lambda\theta-1} \dot{z}$ Here λ is any whole Number greater than 0; r is half of any whole Number whatever.
24	$\frac{\overline{e + fz^\theta}^m z^{\lambda\theta-1} \dot{z}}{\overline{g + hz^\theta}^{m+r}}$ Here either λ or $r - \lambda$ must be a positive whole Number greater than 0.

Transform'd into others.

$$\frac{v^{\lambda-1}}{v-e} \times \frac{v^m \dot{v}}{\theta f^\lambda}.$$

$$v = e + fz^\theta.$$

$$\frac{fz^{\lambda\theta-1} \dot{z}}{p \times e + fz^\theta} - \frac{hz^{\lambda\theta-1} \dot{z}}{p \times g + hz^\theta}.$$

$$p = fg - eh.$$

$$\frac{v^{\lambda-1}}{v-e} \times \left[p + \frac{hv}{f} \right]^r \times \frac{v^m \dot{v}}{\theta f^\lambda}.$$

$$p = g - \frac{eh}{f}.$$

$$v = e + fz^\theta.$$

$$\frac{gv^{\lambda-1}}{gv-e} \times \frac{f - hv^{r-\lambda-1}}{\theta p^{r-1}} v^m \dot{v}.$$

$$p = fg - eh.$$

$$v = \frac{e + fz^\theta}{g + hz^\theta}.$$

Forms.	Fluxions.
25	$\frac{e + fz^{\theta} z^{\lambda\theta-1} \dot{z}}{k + lz^{\theta} \times g + bz^{\theta})^{m+r}}$ Here r is any affirmative whole Number whatsoever.
26	$\frac{z^{\lambda\theta-1} \dot{z}}{e + fz^{\theta} + gz^{2\theta}}$ N. B. This Form fails when $4eg$ is greater than ff.
27	$\frac{e + fz^{\theta} + gz^{2\theta} z^{\lambda\theta-1} \dot{z}}{}$ λ is any positive whole Number greater than 0; m is the half of any whole Number.
28	$\frac{z^{\lambda\theta-1} \dot{z}}{e + fz^{\theta} + gz^{2\theta}}$ Here e and g are affirmative. λ is the half of any affirmative whole Number greater than 1. N. B. This Form fails when ff is greater than $4eg$.

Transform'd.

$$\frac{f - bv^r}{\theta lb^r} \times \frac{av^m \dot{v}}{d + av} + \frac{f - bv^{r-1} bv^m \dot{v}}{\theta lb^r}.$$

$$a = gl - kb.$$

$$b = fg - eh.$$

$$d = fk - el.$$

$$v = \frac{e + fz^\theta}{g + bz^\theta}.$$

$$\frac{g}{p} \times \frac{z^{\lambda\theta-1} \dot{z}}{\frac{f-p}{2} + gz^\theta} - \frac{g}{p} \times \frac{z^{\lambda\theta-1} \dot{z}}{\frac{f+p}{2} + gz^\theta}.$$

$$p = \sqrt{ff - 4eg}.$$

$$\frac{\overline{v - \frac{1}{2}f}^{\lambda-1}}{\theta g^\lambda} \times p + \frac{v^2}{g} \Bigg| \dot{v}.$$

$$p = e - \frac{ff}{4g}.$$

$$v = \frac{1}{2}f + gz^\theta.$$

$$\frac{\overline{v + \frac{1}{2}s}^{2\lambda-2}}{\theta sg^{2\lambda-3}} \times \frac{\dot{v}}{p + v^2} - \frac{\overline{V - \frac{1}{2}s}^{2\lambda-2}}{\theta sg^{2\lambda-3}} \times \frac{\dot{V}}{p + V^2}.$$

$$p = \frac{1}{4}fg + \frac{1}{2}g\sqrt{eg}.$$

$$s = \sqrt{2g\sqrt{eg} - fg}.$$

$$v = -\frac{1}{2}s + gz^{\frac{1}{2}\theta}.$$

$$V = +\frac{1}{2}s + gz^{\frac{1}{2}\theta}.$$

Forms.	Fluxions.
29	$\frac{z^{\lambda\theta-1} \dot{z}}{b + kz^{\theta} \times e + fz^{\theta} + gz^{2\theta}}$ N. B. This Form fails when $4eg$ is greater than ff.
30	$\frac{z^{\lambda\theta-1} \dot{z}}{k + lz^{\theta} \times e + fz^{\theta} + gz^{2\theta}}$ Here λ is any affirmative whole Number greater than 0.
31	$\frac{z^{\lambda\theta-1} \dot{z}}{k + lz^{\theta} \times e + fz^{\theta} + gz^{2\theta}}$ Here e and g are affirmative. λ is the half of any affirmative whole Number greater than 2. N. B. This Form fails when ff is greater than $4eg$.

Transform'd.

$$\frac{gkkz^{\lambda\theta-1}\dot{z}}{cd.b+kz^\theta} + \frac{ggz^{\lambda\theta-1}\dot{z}}{dp.b+gz^\theta} - \frac{ggz^{\lambda\theta-1}\dot{z}}{pc.a+gz^\theta}.$$

$$p = \sqrt{ff-4eg}.$$

$$a = \frac{f-p}{2}.$$

$$b = \frac{f+p}{2}.$$

$$c = ak - gb.$$

$$d = bk - gb.$$

$$\frac{llz^{\lambda\theta-1}\dot{z}}{t.k+lz^\theta} + \frac{\overline{v-\frac{1}{2}f}^{\lambda-1}}{\theta tg^{\lambda-1}} \times \frac{a\dot{v}-lv\dot{v}}{p+v^2}.$$

$$a = gk - \frac{1}{2}lf.$$

$$p = eg - \frac{1}{4}ff.$$

$$t = ell - fkl + gkk.$$

$$v = \frac{1}{2}f + gz^\theta.$$

$$-\frac{lkz^{\lambda\theta-\theta-1}\dot{z}}{t.k+lz^\theta} + \frac{\overline{v+\frac{1}{2}s}^{2\lambda-3}}{2\theta stg^{2\lambda-3}} \times \frac{-2qv\dot{v}+rs\dot{v}}{p+v^2}$$

$$+ \frac{\overline{V-\frac{1}{2}s}^{2\lambda-3}}{2\theta stg^{2\lambda-3}} \times \frac{2qV\dot{V}+rs\dot{V}}{p+V^2}.$$

$$p = \frac{1}{2}g\sqrt{eg} + \frac{1}{4}fg.$$

$$q = l\sqrt{eg} - gk.$$

$$r = l\sqrt{eg} + gk.$$

$$s = \sqrt{2g\sqrt{eg} - fg}.$$

$$t = ell - fkl + gkk.$$

$$v = -\frac{1}{2}s + gz^{\frac{1}{2}\theta}.$$

$$V = +\frac{1}{2}s + gz^{\frac{1}{2}\theta}.$$

Forms.	Fluxions.
32	$\frac{z^{\theta-1} \dot{z}}{k + lz^{\theta} \times e + fz^{\theta} + gz^{2\theta^m}}$ Here m is half of any positive whole Number.
33	$\frac{z^{\lambda\theta-1} \dot{z} \times e + fz^{\theta^m}}{h + kz^{\theta} + lz^{2\theta}}$ Here λ is any affirmative whole Number greater than 0. N. B. This Form fails when $4bl$ is greater than kk.

Transform'd.

$$\frac{\overline{lv - \frac{1}{2}b}^{2m-1}}{-n} \times \frac{\dot{v}}{\overline{ap + av^2}^m}.$$

$$a = ell - fkl + gkk.$$

$$b = fl - 2gk.$$

$$p = eg - \frac{1}{4}ff.$$

$$v = \frac{el - \frac{1}{2}fk + \frac{1}{2}bz^{\theta}}{k + lz^{\theta}}.$$

$$\frac{l \times \overline{v - e}^{\lambda-1}}{\theta p f^{\lambda-1}} \times \frac{v^m \dot{v}}{\frac{k-p}{2} f - le + lv}$$

$$- \frac{l \times \overline{v - e}^{\lambda-1}}{\theta p f^{\lambda-1}} \times \frac{v^m \dot{v}}{\frac{k+p}{2} f - le + lv}.$$

$$p = \sqrt{kk - 4bl}.$$

$$v = e + fz^{\theta}.$$

Forms.	Fluxions.
34	$\frac{z^{\lambda\theta-1} \dot{z} \times \overline{e + fz^{\theta^m}}}{b + kz^{\theta} + lz^{2\theta}}$ Here <i>b</i> and <i>l</i> are affirmative. λ is any positive whole Number greater than 0. m is the half of any affirmative odd Number. N. B. This Form fails when kk is greater than $4bl$, or when a is negative.
35	$\frac{z^{\lambda\theta-1} \dot{z}}{b + kz^{\theta} + lz^{2\theta} \times \overline{e + fz^{\theta^m}}}$ Here <i>b</i> and <i>l</i> are affirmative. λ is any positive whole Number less than $m + 1$. m is the half of any positive odd Number. N. B. This Form will fail when kk is greater than $4bl$, or when a is negative.

Transform'd.

$$\frac{vv - sv + q^{\lambda-1} \times v + \frac{1}{2}s^{2m}}{\theta s f^{\lambda-2} l^{2\lambda+2m-3}} \times \frac{\dot{v}}{p + vv}$$

$$\frac{VV + sV + q^{\lambda-1} \times V - \frac{1}{2}s^{2m}}{\theta s f^{\lambda-2} l^{2\lambda+2m-3}} \times \frac{\dot{V}}{p + VV}.$$

$$a = ff b - e f k + e e l.$$

$$b = f k - 2 e l.$$

$$s = \sqrt{2 l \sqrt{a l} - b l}.$$

$$p = l \sqrt{a l} - \frac{1}{4} s s.$$

$$q = \frac{1}{4} s s - e l l.$$

$$v = -\frac{1}{2} s + l \sqrt{e + f z^{\theta}}.$$

$$V = +\frac{1}{2} s + l \sqrt{e + f z^{\theta}}.$$

$$\frac{-e V V + e s V + q^{\lambda-1} \times V - \frac{1}{2}s^{2m-2\lambda+2}}{\theta s f^{\lambda-2} a^m l^m} \times \frac{\dot{V}}{p + V^2}$$

$$\frac{-e v v - e s v + q^{\lambda-1} \times v + \frac{1}{2}s^{2m-2\lambda+2}}{\theta s f^{\lambda-2} a^m l^m} \times \frac{\dot{v}}{p + v^2}.$$

$$a = ff b - e f k + e e l.$$

$$b = f k - 2 e l.$$

$$s = \sqrt{2 l \sqrt{a l} - b l}.$$

$$p = l \sqrt{a l} - \frac{1}{4} s s.$$

$$q = a l - \frac{1}{4} s s e.$$

$$v = -\frac{1}{2} s + \sqrt{\frac{a l}{e + f z^{\theta}}}.$$

$$V = +\frac{1}{2} s + \sqrt{\frac{a l}{e + f z^{\theta}}}.$$

The Use of the foregoing Table of Fluxions.

1. **W**HEN it is required to find the Fluent of a given Fluxion, by this Table; it must first be found out to what *Form* it belongs. To this End substitute n or θ (as the Case requires) for the numeral Index, and then it may be compared with the several *Forms* in the Table. Or the given Fluxion may (and often must) be *reduced* to another Expression, by dividing it by the highest Power in the Radical, (that is, by taking the highest Power of the variable Quantity out of the Radical) and affecting the other Quantity with it; by which Means the Fluxion will acquire a new Form and Expression, for the Sign of the Index will be changed, (*and this I call reducing the Index*). Then substitute n or θ for the numeral Index, and find what Form in the Table it will agree to.

For Example, let $\frac{\dot{x}}{aa + xx\sqrt{bb - xx}}$ be given

here x^2 will be x^θ , $\theta = 2$, and $\frac{1}{2}\theta = 1$; and $\dot{x}$ or $x^{1-\frac{1}{2}\theta}$

$= x^{\frac{1}{2}\theta - 1} \dot{x}$; therefore the Fluxion is $\frac{x^{\frac{1}{2}\theta - 1} \dot{x}}{aa + x^\theta \sqrt{bb - x^\theta}}$

then comparing this with the *Twenty-third Form* we shall have $\lambda = \frac{1}{2}$, which ought to be a whole Number, and therefore it comes not under this Form. Now reduce the Index by dividing by xx , and we

shall have $\frac{\dot{x}}{x^3 \times 1 + aax^{-2} \sqrt{-1 + bbx^{-2}}}$

$\frac{\dot{x}}{x^{-2-1} \dot{x}}$; where θ will be $= -2$

and the Index $-2 - 1 = \theta - 1$, therefore $\lambda = 1$ and therefore it belongs to Form 23.

2. If the Fluxion, then, is a Binomial, you must first try if the Fluent of either Expression can be had in *finite Terms* by Form 3d or 15th, (which may always be known by the Notes in the second Column); if it cannot, then try whether it can be had in finite Terms by any other of the Forms for Binomials, which will be easy by comparing the Indices. If it fall under the 11th, 12th, 13th, or 14th Form, there will be required two (or perhaps three) Operations, whereof the *first* is always for the Fluent of the *original Fluxion*, and is to be found by some of the first ten Forms. But if it cannot be found in finite Terms by these Forms, then the *last Recourse* is to the 15th, 16th, 17th, or 18th Form, as best suits the Case.

3. Having found to what Form (or Forms) the Fluxion belongs, you have no more to do, but only to write the respective *Values* of the *general Quantities* in the Fluent belonging to that *Form*, and multiply the *Result* by such given Quantities as the given Fluxion was multiply'd into, and you have the *Fluent*.

4. And in compound Binomials, or Trinomials, such as belong to the 21st and following Forms, since these are transform'd into single binomial Fluxions, standing in the third Column; therefore you must proceed with these according to the foregoing Directions for Binomials; and the Fluents of these (single) Binomials being found, will be the Fluents of the Trinomial or compound Binomial Fluxions standing in the second Column, respectively.

5. But there are several Sorts of compound Fluxions which cannot be resolved without some further Reduction. Now there are these two other Ways of reducing a given Fluxion to a different Form or Expression. The first is actually to multiply by any Power of the Quantity under the Vinculum (in any Binomial or Trinomial Surd) and then lessen the Index of that Surd by the same Power if it is in the

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Numerator,

Numerator, or increase it in the Denominator: This alters the Power of the surd Quantity. The other Way is actually to divide by the Quantity under the Vinculum, and then lessen the Index of that Surd by 1 if it be in the Denominator, or increase it by 1 if it is in the Numerator: This alters the Dimension of the simple variable Quantity. And in both you will have the more Terms according as you multiply by a greater Power, or continue your Division the further. But the last Term only will be of a like Form with the given Fluxion, differing only in the Index of the Surd, or of the variable Quantity.

6. Therefore in any Fluxion, but especially in compound Binomials and Trinomials, if there be a Surd in the Numerator you may *lessen* its Dimension at Pleasure (or take it quite out of the Numerator if you *multiply* by some *Power* of the Quantity under the Vinculum, and *lessen* its Index by the *same* Power.

7. Also in any given Fluxion, if the Index of the simple variable Quantity be too high, (as 2θ , 3θ , &c.), or when it is too low or negative, (as -1 , -2θ , &c.) it may be alter'd at Pleasure, by *dividing* by some rational Compound Quantity in the Denominator if there is any, or else by the Quantity under the Vinculum, and *subtracting* 1 from its Index, if it is in the *Denominator*, or adding 1 in the *Numerator*, and continuing the Division to a proper Number of Places.

For Example, if $\frac{\dot{x}\sqrt{bb-xx}}{aa+xx}$ be given; reduce

the Index, and it becomes $\frac{x^{-1}\dot{x}\sqrt{-1+bbx^{-2}}}{1+aa x^{-2}}$, but

neither Way will it agree with any of the Forms for in the first Case $\theta=2$, and $\dot{x}=x^{1-1}\dot{x}=x^{\frac{1}{2}-1}\dot{x}$ which makes $\lambda=\frac{1}{2}$; and in Case 2d, $\theta=-2$, and $x^{-1}\dot{x}=x^{0-1}\dot{x}$, and $\lambda=0$; but λ ought to be a whole Number

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Number. Therefore (by Art. 6.) I multiply both Numerator and Denominator of $\frac{\dot{x}\sqrt{bb-xx}}{aa+xx}$ by

$$\sqrt{bb-xx}, \text{ and it becomes } \frac{bb\dot{x} - x^2\dot{x}}{aa+xx\sqrt{bb-xx}};$$

now the first Term $\frac{bb\dot{x}}{aa+xx\sqrt{bb-xx}}$ comes under the 23d Form, being of the same Form as that in

Art. 1. And the latter Term $\frac{-x^2\dot{x}}{aa+xx\sqrt{bb-xx}}$ being actually divided by $xx+aa$ (according to Art. 7.)

it becomes $\frac{-\dot{x}}{\sqrt{bb-xx}} + \frac{aax\dot{x}}{aa+xx\sqrt{bb-xx}}$, the latter Term (like that Art. 1.) belongs to Form 23d; and the former to Form 10.

Or thus.

Divide $\frac{\dot{x}\sqrt{bb-xx}}{aa+xx}$ by $aa+xx$, and it becomes

$$\frac{\dot{x}\sqrt{bb-xx}}{xx} - \frac{aax\dot{x}\sqrt{bb-xx}}{xx \times aa+xx}. \text{ Reduce the Index}$$

in the latter Term, and it is $\frac{-aax^{-2-1}\dot{x}\sqrt{bb-xx}}{1+aax^{-2}},$

which belongs to the 23d or 24th Form, where $\lambda=1$; and the first Part belongs to Form 12.

By these Operations the given Fluxion is reduced to several Terms; all which (except the last) must be still further reduced if there be Occasion, by repeating the same Method, till at last all the resulting Terms will fall under some or other of the Forms in the Table; as in the Examples here given, and many more will follow afterwards. By these Sorts of Re-

duction, a given Fluxion is prepared for a Solution when it does not fall directly at first, under any of the Forms.

8. In *Form 16*, when the Denominator of any Term happens to ^{be} 0, that particular Term must be found thus; take the *known Part* of the Numerator (that is leaving out the Powers of x), and multiply it into $2.302585 \text{ Log. } x$, and you have that Term. And the said known Part of the Numerator will be the Value of the Capital Letter in the next following Term.

For Example, In finding the Fl: $x^4 \sqrt{1 + \frac{1}{xx}}$

we have this Series $\frac{x^4}{4} + \frac{\frac{1}{2}A \frac{1}{xx}}{2} - \frac{\frac{1}{4}B \frac{1}{xx}}{0} +$
 $\frac{\frac{1}{2}C \frac{1}{xx}}{2} - \frac{\frac{5}{8}D \frac{1}{xx}}{4} \text{ \&c.}$ Instead of the third Term,
 take $-\frac{1}{4}B \times \text{Log. } x$. But $B = \frac{1}{2}A = \frac{1}{2}$ (without x); there-
 fore $-\frac{1}{4}B = -\frac{1}{8} = C$. Then the Series is $\frac{x^4}{4} +$
 $\frac{\frac{1}{2}A \frac{1}{xx}}{2} - \frac{1}{8} \times 2.30258 \text{ Log. } x + \frac{\frac{-1 \times 1}{8 \times 2xx}}{2} - \frac{\frac{5}{8}D \frac{1}{xx}}{4}$
 $\text{\&c.}$

9. Though these Forms contain Variety of Logarithmic Fluents; yet each of them may be changed several Ways into different Quantities. Thus, when a Logarithm is in the Fluent, you may multiply (the *Number* whose Log. is there) and then divide it by some compound Quantity, which you see will make it simpler: Or you may square it and take half the Logarithm: Or you may extract the square Root and take double the Logarithm: Or you may multiply it or divide it by any *given* Quantity: Or make the Numerator and Denominator change Places, and then

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then change the Sign of the Logarithm, &c. And thus you may always find the simplest Expression for the Logarithmic Quantity.

$$\text{Thus, Log: } \frac{x}{y} = \text{Log: } \frac{ax}{ay} = \frac{1}{2} \text{Log: } \frac{xx}{yy} = 2 \text{ Log:}$$

$$\frac{x}{y} = -\text{Log: } \frac{y}{x}, \text{ \&c.}$$

10. And if you have an untractable Fluxion that will answer to none of the Forms; it may sometimes be transformed into others, by Prop. 9, which may then be resolved by the Table.

The following Examples will make the Process very plain.

EXAMPLE I.

To find the Fluent of $\frac{x\dot{x}}{xx - aa}$.

Here $n = 2$, by which expunging the numeral index, the Fluxion will be reduced to $\frac{x^n - 1 \dot{x}}{-aa + x^n}$, which is a Fluxion belonging to the 4th Form. Therefore $\alpha = -aa$, $\beta = 1$, $z = x$, and $\frac{2.3025}{\eta\beta}$

$$\text{Log. of } \frac{x}{\alpha + \beta z^n} = \frac{2.3025}{2} \times \text{Log. } \frac{xx - aa}{x} = 3025851 \text{ Log. } \sqrt{x^2 - a^2}, \text{ the Fluent required.}$$

Ex. 2.

Let the Fluxion $\frac{r\dot{x}}{\sqrt{bb + xx}}$ be given.

Here $n = 2$; whence the Fluxion will be reduced to $\frac{rx^{\frac{1}{2}n-1} \dot{x}}{\sqrt{bb + x^n}}$, a Fluxion of the 9th Form; whence

$$\alpha = bb, \beta = 1, z = x, \text{ and } \phi = 2.30258 \text{ Log. } x + \sqrt{bb + xx}$$

= Fluent of $\frac{\dot{x}}{\sqrt{bb + xx}}$; therefore the Fluent

$$\frac{rx\dot{x}}{\sqrt{bb + xx}} = 2.302585r \times L: x + \sqrt{bb + xx}.$$

Ex. 3.

To find the Fluent of $\frac{bx^{\frac{2}{3}}\dot{x}}{c - ax^{\frac{2}{3}}}$.

Here $n = \frac{2}{3}$, and the Fluxion becomes $c - ax^n$ $bx^{2n-1}\dot{x}$, which belongs to Form the 3d and 11th.

Or rather thus, by taking $x^{\frac{2}{3}}$ out of the Radical, the

Fluxion becomes $\frac{1}{c - ax^{-\frac{2}{3}}} \times bx^{-\frac{1}{3}}\dot{x}$, which belongs to form 15th. Here $\alpha = -a$, $\beta = c$, $\gamma =$

$n = -\frac{2}{3}$, $\mu = -\frac{3}{5}$, $\pi = -\frac{1}{5}$, $\epsilon = \frac{1}{5}$, $\delta =$

Then since $\frac{\pi+1}{n} + \mu = -2$ a negative whole Number, therefore the Fluent will be had in finite Terms

and is = $b \times: -\frac{15x^{\frac{14}{5}}}{14a} - \frac{75cx^{\frac{4}{5}}}{28aa} : \times -a + cx^{-\frac{2}{3}}$

= $-\frac{30abx^{\frac{2}{3}} + 75bc}{28aa} \times c - ax^{\frac{2}{3}}$, the Fluent required.

Ex. 4.

To find the Fluent of $\frac{bz\dot{z}}{z\sqrt{zz - cc}}$.

This belongs to Form the 12th, having by Form the 3d the Fluent of $\frac{bz\dot{z}}{\sqrt{zz - cc}}$; in which Case

$\pi = 1$, $n = 2$, and $-1 = 1 - 2 = \pi - n$, and the

Given Fluxion becomes $\frac{bz^{\pi-\eta}\dot{z}}{\sqrt{-c^2+z^\eta}}$. But since

$\frac{+1}{\eta} = 1$, therefore the Fluent cannot be had by Form the 12th. Wherefore I reduce it to

$\frac{bz^{-2}\dot{z}}{\sqrt{1-ccz^{-2}}}$, which is a Fluxion of the 10th Form,

where $\eta = -2$, $\alpha = 1$, $\beta = -cc$; and the Fluent

$= -\frac{b}{c} \times .017453$ Degrees of the Arch whose Sine

$\frac{c}{z}$.

Ex. 5.

Let the Fluent of $\frac{a\dot{x}}{x}\sqrt{2ax+xx}$, or $ax^{\frac{1}{2}-1}\dot{x}\sqrt{2a+x}$ be required.

This belongs to the 9th and 13th Forms. To get

the Fluent of $\frac{x^{\frac{1}{2}-1}\dot{x}}{\sqrt{2a+x}}$ by Form 9th; here

$= 2a$, $\beta = 1$, $z = x$, $\eta = 1$: Whence the Fluent

$= 2.3025 \text{Log}: a+x+\sqrt{2ax+xx} = \phi$. The given

Fluxion therefore would become $ax^\pi\dot{x} \times \overline{2a+x}^{\mu+\tau}$.

Where $\pi = -\frac{1}{2}$, $\mu = -\frac{1}{2}$, $\tau = 1$, $\alpha = 2a$, $\beta = 1$,

$\gamma = 2a+x$, $\delta = 0$: And the Fluent

of $x^{-\frac{1}{2}}\dot{x}\sqrt{2a+x}$ is $= a\phi + x^{\frac{1}{2}}\sqrt{2a+x}$, by Form 13.

And the Fluent of $ax^{-\frac{1}{2}}\dot{x}\sqrt{2a+x} = a\phi + \sqrt{2ax+xx}$.

Ex.

Ex. 6.

To find the Fluent of $\frac{bz^2 \dot{z}}{aa - \frac{aa}{bb} z^2}$.

This will be had by Form the 6th and 11th.
To find the Fluent of $\frac{\dot{z}}{aa - \frac{aa}{bb} z^2}$ by Form 6th

Here $\alpha = aa$, $\beta = \frac{aa}{bb}$, $n = 2$, then the Fluent is

$$\frac{b}{2aa} \times 2.3025 \text{ Log. } \frac{a + \frac{a}{b} z}{a - \frac{a}{b} z} = \frac{b}{aa} \times 2.3025 \times \text{Log.}$$

$$\sqrt{\frac{b+z}{b-z}} = \phi. \text{ Then by Form 11th, } \alpha = aa, \beta = \frac{aa}{bb}, \mu = -1, \pi = 0, \lambda = 1, \gamma = 1, \epsilon = 3, \delta =$$

$$\text{whence the Fluent of } \frac{z^2 \dot{z}}{aa - \frac{aa}{bb} z^2} \text{ is } = \frac{-aa \times 1}{-\frac{aa}{bb} \times 1}$$

$$+ \frac{bbz}{-aa} = bb\phi - \frac{bb}{aa} z; \text{ and consequently the Flu}$$

$$\text{ent of } \frac{bz^2 \dot{z}}{aa - \frac{aa}{bb} z^2} \text{ is } = b^3 \phi - \frac{b^3}{aa} z = \frac{b^4}{aa}$$

$$2.3025 \text{ Log. } \sqrt{\frac{b+z}{b-z}} : - \frac{b^3}{aa} z.$$

Ex. 7.

To find the Fluent of $b\dot{x} \sqrt{\frac{4a-3x}{a-x}}$.

This by Form 23d is transformed into $b \times \overline{a+3v^{\frac{1}{2}}} \times \frac{v^{-\frac{1}{2}}\dot{v}}{-1}$. Where $\lambda = 1$, $e+fz^{\theta} = a-x=v$, $g+bz^{\theta} = 4a-3x$, $p = a$: By Form the 9th the Fluent of $\frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{a+3v}}$ is $\frac{2}{\sqrt{3}} \times 2.30258 \times \text{Log.} \sqrt{3v} + \sqrt{a+3v} = \phi$. And by Form 13th, Fluent of $-bv^{-\frac{1}{2}}\dot{v} \sqrt{a+3v}$ is $= -\frac{ba}{2}\phi - bv^{\frac{1}{2}}\sqrt{a+3v} = -\sqrt{a-x} \times 4a-3x - \frac{2.30258ba}{\sqrt{3}} \times \text{Log.} \sqrt{3a-3x} + \sqrt{4a-3x}$.

Ex. 8.

To find the Fluent of $\frac{x^{-1}\dot{x}\sqrt{aa-xx}}{bb + \frac{b}{a}xx}$.

This by Division is reduced to $\frac{x^{-1}\dot{x}}{bb} \sqrt{aa-xx} - \frac{xx}{bb + \frac{b}{a}xx} \times \frac{\sqrt{aa-xx}}{b}$. The Fluent of $\frac{x^{-1}\dot{x}}{bb} \sqrt{aa-xx}$ (by Form 9th and 13th) is $= \frac{\sqrt{aa-xx}}{bb} - \frac{2.30258a}{bb} \times \text{Log.} \frac{a + \sqrt{aa-xx}}{x}$. And by Form the 23d, $\frac{xx\sqrt{aa-xx}}{bb + \frac{b}{a}xx}$ is transformed to $p + \frac{b}{f}v^{-1} \times \frac{v^{\frac{1}{2}}\dot{v}}{bf} =$

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$$= \frac{-\frac{1}{2}v^{\frac{1}{2}}\dot{v}}{bb + ba - \frac{b}{a}v} : \text{For } g + bz^{\theta} = bb + \frac{b}{a}xx, e + f$$

$$= aa - xx = v, p = bb + ba, \theta f = -2, m = \frac{1}{2}, r = -1$$

$$\lambda = 1. \text{ Now by Form the 6th, the Fluent of } \frac{v^{-\frac{1}{2}}\dot{v}}{p - \frac{b}{a}v}$$

$$= 2.30258 \sqrt{\frac{a}{pb}} \times \text{Log. of } \frac{p + \frac{b}{a}v + 2\sqrt{\frac{pbv}{a}}}{p - \frac{b}{a}v} =$$

$$\text{And by Form 11th the Fluent of } \frac{\frac{1}{2ba} \times \frac{v^{\frac{1}{2}}\dot{v}}{p - \frac{b}{a}v}}{p - \frac{b}{a}v}$$

$$= \frac{p}{2bb} \phi - \frac{\sqrt{v}}{bb} : \text{Whence the Fluent of } \frac{x^{-1}\dot{x}\sqrt{aa-xx}}{bb + \frac{b}{a}xx}$$

$$\text{is } = \frac{\sqrt{aa-xx}}{bb} - \frac{\sqrt{v}}{bb} + \frac{2.30258a}{bb} \text{Log. } \frac{a - \sqrt{aa-xx}}{x}$$

$$+ \frac{p}{2bb} \phi = \frac{2a \times 2.30258 \text{ Log. } \frac{a - \sqrt{v}}{x} : + p}{2bb}$$

Ex. 9.

The Fluent of $bbx^{-2}\dot{x} \times \frac{\overline{a+x^{\frac{3}{2}}}}{\sqrt{ab+ax}}$ is required.

Here $\theta = 1$, and the Fluxion would become

$$bbx^{-\theta-1}\dot{x} \times \frac{\overline{a+x^{\theta\frac{3}{2}}}}{\sqrt{ab+ax^{\theta}}}, \text{ therefore the given Fluxion}$$

$$\text{is reduced by Multiplication to } \frac{bbaax + 2bbax\dot{x} + bbx^{\frac{3}{2}}}{xx\sqrt{ab+ax} \times \sqrt{a+x^{\frac{3}{2}}}}$$

$$= \frac{bb\dot{x}}{\sqrt{ab+ax}\sqrt{a+x}} + \frac{2bbax^{-2}\dot{x}}{\sqrt{a+abx^{-1}}\sqrt{1+ax^{-1}}} + \frac{bbax^{-3}\dot{x}}{\sqrt{a+abx^{-1}}\sqrt{1+ax^{-1}}}; \text{ each Term whereof belongs}$$

to the 23d Form. Let $e+fz^{\theta}=a+x=v$, $g+bz^{\theta}=b+ax$, $\theta=1$, $r=-\frac{1}{2}$, $m=-\frac{1}{2}$, $p=ab-aa$, then

$$\frac{bb\dot{x}}{\sqrt{ab+ax}\sqrt{a+x}} = \frac{bbv^{-\frac{1}{2}}\dot{v}}{\sqrt{p+av}}, \text{ whose Fluent by}$$

Form the 9th will be $\frac{2bb}{\sqrt{a}} \times 2.30258 \text{ Log:}$

$$\sqrt{av} + \sqrt{p+av} : = \frac{2bb}{\sqrt{a}} \times 2.30258 \text{ Log:}$$

$$\sqrt{aa+ax} + \sqrt{ab+ax} : = 2.302585 \times \frac{2bb}{\sqrt{a}} \times \text{Log:}$$

of $\sqrt{a+x} + \sqrt{b+x} : = \frac{2bb}{\sqrt{a}} A$, the Fluent of the first Term.

Again for the 2d and 3d Terms: Make $e+fz^{\theta}=a+ax^{-1}=v$, $g+bz^{\theta}=a+abx^{-1}$, $\theta=-1$, $p=a-b$, $\lambda=1$ in the second Term, and $\lambda=2$ in the third Term,

which therefore will be transform'd into $-\frac{2bbv^{-\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$

and $\frac{bbv^{-\frac{1}{2}}\dot{v} - bbv^{\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$, whose Sum is $-\frac{bbv^{-\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$

$\frac{bbv^{\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$. By Form the 9th the Fluent of $\frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$

$$= \frac{2}{\sqrt{b}} \times 2.3025 \text{ Log: } \sqrt{bv} + \sqrt{p+bv} : = 2.30258$$

$$\times \frac{2}{\sqrt{b}} \times \text{Log: } \sqrt{\frac{ba+bx}{x}} + \sqrt{\frac{ba+ax}{x}} : = \frac{2}{\sqrt{b}} B. \text{ And}$$

by Form the 11th, the Fluent of $\frac{v^{\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$ is $= \frac{-p}{b\sqrt{b}}$
 $+\frac{v^{\frac{1}{2}}}{b}\sqrt{p+bv}$, multiply the Sum of these Fluents by

$-bb$, and you will have $\frac{-2bb+bp}{\sqrt{b}} \times B - bv^{\frac{1}{2}}\sqrt{p+bv}$

$\frac{ba-3bb}{\sqrt{b}}B - \frac{b}{x}\sqrt{ab+ax}\sqrt{a+x}$, to which add the

first Term $\frac{2bb}{\sqrt{a}}A$, and you will have $\frac{2bb}{\sqrt{a}}A$

$\frac{ba-3bb}{\sqrt{b}}B - \frac{b}{x}\sqrt{ab+ax}\sqrt{a+x}$, the Fluent is

quired.

Ex. 10.

To find the Fluent of $\frac{\dot{z}}{4bz}\sqrt{a^2b^2-a^3z+16b^2z^2}$

The given Fluxion will be reduced to $\frac{\dot{z}}{4bz}$

$$\frac{a^2b^2-a^3z+16b^2z^2}{\sqrt{a^2b^2-a^3z+16b^2z^2}} = \frac{a^2bz^{-2}\dot{z}}{4\sqrt{16b^2-a^3z^{-1}+a^2bz^{-2}}}$$

$$= \frac{a^2\dot{z}}{4b\sqrt{a^2b^2-a^3z+16b^2z^2}} + \frac{a^2bz^{-2}\dot{z}}{4bz\dot{z}\sqrt{a^2b^2-a^3z+16b^2z^2}}$$

each of these Terms belongs to the 27th Form.

For the first Term, $v = -\frac{1}{2}a^3 + a^2b^2z^{-1}$, $p = 16b^2$

$-\frac{a^4}{4bb}$, $\lambda = 1$, $m = -\frac{1}{2}$, $\theta = -1$, and it will be

$$\text{transformed into } \frac{-\dot{v}}{4b\sqrt{p+\frac{vv}{g}}} = \frac{-a\dot{v}}{4\sqrt{pg+vv}}$$

whose Fluent by Form the 9th will be $-\frac{a}{4}$

$$2.30258 \text{ Logar: } v + \sqrt{pg+vv} = \frac{-a}{4} \times 2.30258$$

$$\text{Log: } \frac{a^2b^2}{z} - \frac{1}{2}a^3 + \frac{ab}{z}\sqrt{a^2b^2-a^3z+16b^2z^2}.$$

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Again for the other two Terms, here $v = -\frac{1}{2}a^3 + 6b^2z$, $p = a^2b^2 - \frac{a^6}{64b^2}$, $m = -\frac{1}{2}$, $\theta = 1$, $\lambda = 1$ in the second Term, and $= 2$ in the third: And these two Terms by Form the 27th will be transformed into

$$\frac{-a^3\dot{v}}{6bb\sqrt{pg+vv}} \text{ and } \frac{v\dot{v}}{16bb\sqrt{pg+vv}} + \frac{a^3\dot{v}}{32bb\sqrt{pg+vv}}$$

whose Sum is
$$\frac{v\dot{v}}{16bb\sqrt{pg+vv}} - \frac{a^3\dot{v}}{32bb\sqrt{pg+vv}};$$

whose Fluent (by Form the 9th)
$$= \frac{\sqrt{pg+vv}}{16bb} - \frac{a^3L}{32bb}$$

Log. of: $v + \sqrt{pg+vv} = \frac{\sqrt{a^2b^2 - a^3z + 16bbz^2}}{4b}$

$$- 2.302585 \times \frac{a^3}{32bb} \times \text{Log: } -\frac{1}{2}a^3 + 16bbz + 4b$$

$$\sqrt{a^2b^2 - a^3z + 16bbz^2} : \text{Whence the Fluent of } \frac{\dot{z}}{4bz}$$

$$\sqrt{a^2b^2 - a^3z + 16bbz^2} \text{ is } = \frac{1}{4b} \sqrt{a^2b^2 - a^3z + 16bbz^2}$$

$$- \frac{a}{4} \times 2.302585 \text{ Log: } \frac{a^2b^2}{z} - \frac{1}{2}a^3 + \frac{ab}{z} \times$$

$$\sqrt{a^2b^2 - a^3z + 16bbz^2} : - \frac{a^3}{32bb} \times 2.302585 \text{ Log: } 16bbz$$

$$- \frac{1}{2}a^3 + 4b\sqrt{a^2b^2 - a^3z + 16bbz^2}.$$

I shall add a few more Examples, chiefly to illustrate the METHOD of REDUCTION.

Ex. II.

Let $\frac{dz^{\frac{3}{2}-1}\dot{z}}{e+fz}$ be proposed: Divide by z and it will

be reduced to $\frac{dz^{-\frac{2}{2}-1}\dot{z}}{f+ez^{-1}}$. This belongs to Form 7th,

7th, where $n = -1$, $\lambda = 5$, $\delta = 2$, $x = \sqrt[5]{\frac{e}{fz}}$, $K = 36$

$a = .587785$; $b = .951056$; $c = 0$, and $s = .809017$
 $t = -.309017$; $u = -1$; where always observe that
 the Cosines of Arches above 90° are negative Numbers: then will be found P, Q, R in Degrees; and

the Logarithms of $\sqrt{1 - 2sx + xx}$, $\sqrt{1 - 2tx + xx}$,
 $\sqrt{1 - 2ux + xx}$, for which Logarithms put S, T, U,

Then (by Case 6th) the Fluent is $\frac{d}{-f^{\frac{3}{5}}e^{\frac{2}{5}}} \times \text{int}$

$-.309017 \times 2LS + .951056 \times 2NP + .809017 \times 2L$
 $-.587785 \times 2NQ - LV.$

Ex. 12.

Given $x^{-3}\dot{x} \times \overline{aa - x^2}^{\frac{2}{3}}$, this by Form 21st (putting

$v = aa - x^2$) is transformed into $\overline{v - aa}^{-2} \times \frac{v^{\frac{2}{3}}\dot{v}}{-2}$

$-\frac{1}{2} \times \frac{v^{\frac{2}{3}}\dot{v}}{-aa + v^2}$; which belongs to Form 7th and
 14th.

Ex. 13.

Let $\frac{x^{-1}\dot{x} \times \overline{aa + xx}^{\frac{3}{2}}}{\overline{abb - bxx}^{\frac{3}{2}}}$ be given; this is reduced to

$\frac{aax^{-1}\dot{x} \times \overline{aa + xx}^{\frac{1}{2}}}{\overline{abb - bxx}^{\frac{3}{2}}} + \frac{xx\dot{x} \times \overline{aa + xx}^{\frac{1}{2}}}{\overline{abb - bxx}^{\frac{3}{2}}}$, both

which Terms belong to Form 24.

Ex. 14.

Let $\frac{x^2 \dot{x} \times \overline{a+x}^{\frac{1}{3}}}{\overline{b+x} \times \overline{ab-bx}^{\frac{4}{3}}}$ be proposed; Divide by

$+b$, and it is reduced to $\frac{x \dot{x} \times \overline{a+x}^{\frac{1}{3}}}{\overline{ab-bx}^{\frac{4}{3}}} - \frac{b \dot{x} \times \overline{a+x}^{\frac{1}{3}}}{\overline{ab-bx}^{\frac{4}{3}}}$

$\frac{bb \dot{x} \times \overline{a+x}^{\frac{1}{3}}}{\overline{b+x} \times \overline{ab-bx}^{\frac{4}{3}}}$. The two first Terms belong

Form 24th, and the last to Form 25th.

Ex. 15.

Let $\frac{y^{-2} \dot{y}}{a+y \sqrt{ab+cy-yy}}$ be given; Divide by

$+y$, and it will be reduced to $\frac{y^{-2} \dot{y}}{a \sqrt{ab+cy-yy}}$

$\frac{y^{-1} \dot{y}}{aa \sqrt{ab+cy-yy}} + \frac{\dot{y}}{aa \times a+y \sqrt{ab+cy-yy}}$;

the last Term belongs to the 32d Form, and (by

reducing the Index) the two first Terms become

$\frac{y^{-3} \dot{y}}{\sqrt{-1+cy^{-1}+aby^{-2}}} - \frac{y^{-2} \dot{y}}{aa \sqrt{-1+cy^{-1}+aby^{-2}}}$;

with which belong to the 27th Form.

Ex. 16.

Given $\frac{d\dot{x} \sqrt{aa-bx+xx}}{a-x}$; this (by Multipli-

cation) is reduced to $\frac{daa \dot{x}}{a-x \sqrt{aa-bx+xx}} +$

$\frac{dx^2 \dot{x}}{dx^2 \dot{x}}$

$\frac{dx^2 \dot{x} - bdx\dot{x}}{a - x\sqrt{aa - bx + xx}}$; the first Term belongs

Form 32. And dividing the rest by $-x + a$, the

are reduced to $\frac{-dx\dot{x}}{\sqrt{aa - bx + xx}} + \frac{bd - ad \times \dot{x}}{\sqrt{aa - bx + xx}}$

+ $\frac{aad - abd \times \dot{x}}{a - x\sqrt{aa - bx + xx}}$. The last belongs to Form

32, and the two former are Fluxions of the 2^d Form.

Ex. 17.

Given $\frac{x^{-\frac{5}{3}} \dot{x} \times \overline{ab - xx}^{\frac{1}{3}}}{a^4 - 3aax^2 + x^4}$; this (by reduc-

the Index) becomes $\frac{x^{-5} \dot{x} \times -1 + abx^{-2}^{\frac{1}{3}}}{1 - 3aax^{-2} + a^4x^{-4}}$, which

belongs to Form 33d.

Ex. 18.

Let $\frac{z^{-1} \dot{z} \sqrt{aa - zz}}{a^4 - aaz^2 + z^4}$ be proposed; because

is greater than a^4 , it cannot be resolved by the 3^d Form, therefore I divide by $a^4 - aaz^2 + z^4$, which

reduces it to $\frac{z^{-1} \dot{z} \sqrt{aa - zz}}{a^4} + \frac{z \dot{z} \sqrt{aa - zz}}{aa \times a^4 - a^2 z^2 + z^4}$

- $\frac{z^1 \dot{z} \sqrt{aa - zz}}{a^4 \times a^4 - a^2 z^2 + z^4}$; the two last Terms belong

to Form the 34th; and the first is further reduced

$\frac{z^{-2} \dot{z}}{aa \sqrt{1 - aaz^{-2}}} - \frac{z \dot{z}}{a^4 \sqrt{a^2 - z^2}}$; which belong

to Forms 10th and 3d.

Ex. 19.

Let there be given $\frac{x\dot{x}}{aa-ax+xx\sqrt{aa-ax}}$; this is re-

duced to $\frac{aax\dot{x}}{aa-ax+xx\sqrt{aa-ax}} - \frac{ax^2\dot{x}}{aa-ax+xx\sqrt{aa-ax}}$;

the former Term belongs to Form the 35th; and the

latter by Division is further reduced to $\frac{-a\dot{x}}{aa-xx} +$

$\frac{a^2\dot{x} - a^2xx\dot{x}}{(aa-ax+xx\sqrt{aa-ax})^{\frac{3}{2}}}$; the first belongs to the

binomial Forms, and the rest to Form 35; where observe that the Part first found (belonging to Form 35) destroys a Part of this last Fluxion; and then the

whole is reduced to this, $\frac{a^3\dot{x}}{(aa-ax+xx\sqrt{aa-ax})^{\frac{3}{2}}}$

$-\frac{a\dot{x}}{(aa-ax)^{\frac{1}{2}}}$.

Ex. 20.

Given $\frac{z^{rn-1}\dot{z}}{a+bz^n+cz^{2n}+dz^{3n}}$, extract one Root b

of the Denominator (supposed to be put = 0), then

will $b-z^n=0$, by this divide the said Denominator,

and let the Quotient be $e+fz^n+gz^{2n}$; then the given

Fluxion is reduced to this $\frac{z^{rn-1}\dot{z}}{b-z^n \times e + fz^n + gz^{2n}}$,

which belongs to the 29th, 30th, or 31st Form.

Ex. 21.

There is given $\frac{z^{rn-1}\dot{z}}{a+bz^n+cz^{2n}+dz^{3n}+ez^{4n}}$, extract
two Roots a, b of the Denominator; then you have
 $a - z^n = 0$, and $b - z^n = 0$, divide the Deno-
minator by $a - z^n \times b - z^n$, and let the Quotient be
 $e + fz^n + gz^{2n}$. Then the given Fluxion becomes

$$\frac{z^{rn-1}\dot{z}}{a - z^n \times b - z^n \times e + fz^n + gz^{2n}} = \frac{z^{rn-1}\dot{z}}{s \times b - z^n \times e + fz^n + gz^{2n}}$$

$$- \frac{z^{rn-1}\dot{z}}{s \times a - z^n \times e + fz^n + gz^{2n}} \quad (\text{putting } a - b = s,) \text{ both}$$

which Terms belong to the 29th, 30th, or 31st
Form.

If the given Denominator has no Roots, yet
may be divided into two Quadratics by the Rules of

Algebra, as $\frac{z^{rn-1}\dot{z}}{p + qz^n + z^{2n} \times r + sz^n + z^{2n}}$; which
may be transformed to Trinomials, as in Ex.
Prop. 9.

Ex. 22.

Given $\frac{z^{rn-1}\dot{z}}{a+bz^n+kz^{2n} \times e + fz^n + gz^{2n}}$, here r and
 $2m$ must be Integers; put $p = \sqrt{bb - 4ak}$, and
the Fluxion is resolved into these two Terms,

$$\frac{z^{rn-1}\dot{z}}{\left(\frac{b-p}{2} + kz^n \times e + fz^n + gz^{2n}\right)^m} - \frac{k}{p} \times \frac{z^{rn-1}\dot{z}}{\left(\frac{b+p}{2} + kz^n \times e + fz^n + gz^{2n}\right)^m}$$

then dividing by the Denominators $\frac{b-p}{2} + kz^n$, and

$\frac{+p}{2} + kz^n$, if there be Occasion; and the resulting Terms will belong to the 32d Form, and perhaps to some others of the foregoing Forms.

Ex. 23.

To find the Fluent of $\frac{x^{\delta-1}\dot{x}}{e \pm fx^\lambda + x^{2\lambda}}$, where δ, λ are Integers; and $4e > ff$.

Transform it into this $x^{\delta-1}\dot{x} \times \frac{K + kx}{1 + qx + xx} + \frac{L + lx}{+rx + xx} + \text{Ec.}$ by Ex. 14. Prop. IX. and the Fluent of the several Terms will be found by Form 28.

SCHOLIUM.

Although the Construction of many of the foregoing Forms depends on several Things that have not yet been delivered; yet it may be proper in this place to give a short Account thereof, that the Reader may not be intirely ignorant of them, which he will be better understand after he has read such Parts of the following Book as they are founded on.

1. It is plain by Prop. III. that the Fluxions in Form 1, 3, 19 and 20 belong to the Fluents there assigned.

2. Form the 16th is calculated in Example 13, Prop. X. and Form the 18th in Example 35. Likewise the Fluent in Form the 15th is found by Prop. X. assuming $Az^{\pi+1} + Bz^{\pi+\pi+1} + Cz^{\pi+2\pi+1} \text{ Ec. } \times$

$+ \beta z^{\pi\mu+1} = \text{Fluent of } \sqrt{\alpha + \beta z^{\pi}}^{\mu} z \dot{z}.$

3. By Prob. II. Sect. II. the Fluxion of any Quantity divided by that Quantity is the Fluxion of the Hyperbolic Logarithm of that Quantity; and that the

Number 2.302585 reduces the common to the hyperbolic Logarithms. Now if the Fluents in Form 2, 4, 6 and 9th be put into Fluxions according to the afore said Rule, it will appear that they will respectively produce the Fluxions in these Forms.

4. In Example 3d, Prob. X. Sect. II. it is shewn, that the hyperbolic Space between the Assymptotes

is = Fluent of $\frac{ab}{a+x} \dot{x}$, in which if you write RR for ab , Rz (or z) for $a+x$, and R $\dot{z}$ (or $\dot{z}$) for $\dot{x}$, you will have RR $\frac{\dot{z}}{z}$, whose Fluent is the hyperbolic Space

in Form the 2d. Likewise if you write RR for $\frac{\alpha}{\beta} + z^n$ for $a+x$, and $n z^{n-1} \dot{z}$ for $\dot{x}$, you will have

$\frac{nRR\beta z^{n-1}\dot{z}}{\alpha + \beta z^n}$ as in Form the 4th. And lastly, write

RR for ab , and $\sqrt{\alpha} + \sqrt{\beta z^n}$ for $a+x$, and $\sqrt{\beta} \times \frac{1}{2} n z^{\frac{n}{2}-1} \dot{z}$ for $\dot{x}$; and likewise write $\sqrt{\alpha} - \sqrt{\beta z^n}$ for $a+x$, and $-\sqrt{\beta} \times \frac{1}{2} n z^{\frac{n}{2}-1} \dot{z}$ for $\dot{x}$, and the Difference

of these Fluxions will be $\frac{nRR\sqrt{\alpha\beta} \times z^{\frac{n}{2}-1} \dot{z}}{\alpha - \beta z^n}$, in Form the 6th.

5. By Ex. 9. Prob. X. Sect. II. the hyperbolic Sector is equal to the Fluent of $\frac{bba}{2} \times \frac{\dot{t}}{bb - tt}$,

which writing R for a and b , R $\sqrt{\frac{\beta z^n}{\alpha}}$ for t , and

$\frac{n}{2} R \sqrt{\frac{\beta}{\alpha}} \times z^{\frac{n}{2}-1} \dot{z}$ for $\dot{t}$; you'll have

$\frac{nRR\sqrt{\alpha\beta} \times z^{\frac{n}{2}-1} \dot{z}}{4 \times \alpha - \beta z^n}$, as in the 6th Form.

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6. And by the same Example, the hyperbolic Sector is = Fluent of $\frac{aby}{2\sqrt{ab+yy}}$ = (writing R

for a and b) $\frac{RR\dot{y}}{2\sqrt{RR+yy}}$; now if you write $R\sqrt{\frac{\beta z^n}{\alpha}}$

for y, and $\frac{1}{2}R\sqrt{\frac{\beta}{\alpha}} \times z^{\frac{1}{2}n-1} \dot{z}$ for $\dot{y}$, you will have

$\frac{RR\sqrt{\beta} \times z^{\frac{1}{2}n-1} \dot{z}}{4\sqrt{\alpha + \beta z^n}}$; in like manner if you put

$\frac{R}{2}\sqrt{\alpha\beta z^n + \beta^2 z^{2n}} = y$, or $\frac{2R}{-\alpha}\sqrt{\alpha\beta z^n + \beta^2 z^{2n}} =$

when α is a negative Quantity, or $\frac{R}{\sqrt{-\alpha}}\sqrt{\alpha + \beta z^n}$

for y, when α is negative, and expunge y and $\dot{y}$, you will get the Fluxion of the 9th Form any of these ways.

7. Since the Length of any Arch of a Circle (whose radius is R) is $= \frac{3.1416R}{180} \times$ Degrees in that Arch;

Length of the Arch $= .017453R \times$ Degrees in that Arch: And by Example the 5th, Prob. VIII.

Act. II. The Arch = Fluent of $\frac{RR\dot{t}}{RR+tt}$; if you

write $R\sqrt{\frac{\beta z^n}{\alpha}}$ for t, and $\frac{\eta R}{2}\sqrt{\frac{\beta}{\alpha}} \times z^{\frac{1}{2}\eta-1} \dot{z}$ for $\dot{t}$,

you will have $\frac{\eta R\sqrt{\alpha\beta} \times z^{\frac{1}{2}\eta-1} \dot{z}}{2 \times \alpha + \beta z^n}$, whose Fluent

therefore is = that Arch $= .017453R \times$ Degrees in

the Arch $= \frac{2 \text{ Sectors}}{R}$; from all which, the Con-

struction of Form the 5th will easily appear.

8. In

8. In like Manner by the same Problem, the Arc

= Fluent of $\frac{Ry}{\sqrt{RR - yy}}$, in which writing for

$R\sqrt{\frac{\beta z^n}{\alpha}}$, or $\frac{2R}{\alpha}\sqrt{\alpha\beta z^n - \beta^2 z^{2n}}$, and the Fluxion

for y , you will get $\frac{nR\sqrt{\beta}}{2} \times \frac{z^{\frac{n}{2}-1} \dot{z}}{\sqrt{\alpha - \beta z^n}}$

and $nR\sqrt{\beta} \times \frac{z^{\frac{n}{2}-1} \dot{z}}{\sqrt{\alpha - \beta z^n}}$, from which all the other

Varieties in Form the 10th are easily deduced.

9. Form the 11th is gradually calculated by Cor. 2 Prop. IV. where it is demonstrated that the Fluent of

$$\overline{\alpha + \beta z^n}^{\mu} z^{p+n} \dot{z} \text{ is } = \frac{z^{p+1} \times \overline{\alpha + \beta z^n}^{\mu+1} - \overline{p+1} \times \overline{\alpha + \beta z^n}^{\mu}}{\mu n + n + p + 1 : \times \beta}$$

in which writing π for p , ϕ for A , γ for $\overline{\alpha + \beta z^n}^{\mu+1}$

you will have the Fluent of $\overline{\alpha + \beta z^n}^{\mu} z^{\pi+n} \dot{z}$; again

writing $\pi+n$ for p , Fluent of $\overline{\alpha + \beta z^n}^{\mu} z^{\pi+2n} \dot{z}$ (not

found) for A ; you will have the Fluent of

$\overline{\alpha + \beta z^n}^{\mu} z^{\pi+2n} \dot{z}$; again, writing $\pi+2n$ for p , and the

Fluent of $\overline{\alpha + \beta z^n}^{\mu} z^{\pi+2n} \dot{z}$ (last found) for A , you

get the Fluent of $\overline{\alpha + \beta z^n}^{\mu} z^{\pi+3n} \dot{z}$, and so on.

In like Manner the 12th Form is calculated from

Cor. 3. Prop. IV. where the Fluent of $\overline{\alpha + \beta z^n}^{\mu} z^p \dot{z}$

$\frac{z^{p+1} \gamma - \mu n + n + p + 1 \times \beta B}{p + 1 \times \alpha}$. And by a like Pro

cesses the 13th and 14th Forms are gradually calculated

from Cor. 4 and 5, Prop. IV. These four Forms

might have been differently express'd from what the

in the Table. For Example, Form the 11th may be expressed thus; let $e = \pi + 1$, $d = \pi + 1 + \eta$

$\mu\eta$, then the Fluent of $\overline{\alpha + \beta z^n}^\mu z^{\pi + \lambda\eta} \dot{z} =$

$\frac{1}{\alpha} \times e \times \overline{e + \eta} \times \overline{e + 2\eta} \times \overline{e + 3\eta} \times \mathcal{E}c. \text{ to } \lambda \text{ Places}$

$\times \frac{1}{\beta} \times d \times \overline{d + \eta} \times \overline{d + 2\eta} \times \overline{d + 3\eta} \times \mathcal{E}c. \text{ to } \lambda \text{ Places} \times \phi$

$$\frac{z^\gamma}{e\alpha\phi} A - \frac{d\beta z^n}{e + \eta \cdot \alpha} B - \frac{d + \eta \cdot \beta z^n}{e + 2\eta \cdot \alpha} C - \frac{d + 2\eta \cdot \beta z^n}{e + 3\eta \cdot \alpha} D - \mathcal{E}c.$$

continued to $\lambda + 1$ Places; where A, B, C, $\mathcal{E}c.$ are the first, second, third, $\mathcal{E}c.$ Terms with their Signs,

$=$ Fluent of $\overline{\alpha + \beta z^n}^\mu z^{\pi} \dot{z}$, $\gamma = \overline{\alpha + \beta z^n}^{\mu+1}$.

10. The 17th Form is derived from the 11th; for when z and γ are 0, all the Terms vanish except the first, in which taking 0, 1, 2, 3, $\mathcal{E}c.$ Terms (for each of the Terms in the Quantity $e + fz^n + gz^{2n} \mathcal{E}c.$) and multiplying each by its respective Coefficient, and the Sum of all by ϕ , you will at last obtain this Form.

11. The Fluxions in the 21st and following Forms are transform'd by Prop. IX. and their Truth will appear by the bare Substitution of the Quantities herein contained.

12. There still remain the 7th and 8th Forms, whose Calculus is something more difficult: These I investigate after the following manner. Let T, U, W, $\mathcal{E}c.$ stand for the Quantities $\sqrt{1 - 2sx + xx}$,

$\sqrt{1 - 2tx + xx}$, $\sqrt{1 - 2ux + xx}$, $\mathcal{E}c.$ then the

Fluxion of $L \times \text{Log. T}$ (or the hyperbolic Log. T)

will be $\frac{-s\dot{x} + x\dot{x}}{1 - 2sx + xx}$, and the like for U, W, $\mathcal{E}c.$ as

plain from what is delivered in Art. 3. Again, the Fluxion of NP (as will appear by Art. 8.) is

$\frac{a\dot{x}}{1 - 2sx + xx}$, and so for the rest NQ, NR, $\mathcal{E}c.$

Lastly,

Lastly, we must take for granted (for I shall not
to demonstrate it here) that the Product of all
Squares of T, U, W, &c. if λ is an even Number
or the last drawn into the Squares of all the rest,
 λ is an odd Number, is always equal to $1 + x$.
These Things premised, let us investigate any

Case; as for Example, to find the Fluent of $\frac{x}{1+x^3}$

here $\lambda = 3$ and $\frac{180}{3} = 60$: Here $b, c, u, \&c. =$

and $t = -1$, and therefore $\sqrt{1-2tx+xx} = 1+x$
and all Arches above 180 are excluded; hence
assume $LA \times \text{Log. } \sqrt{1-2sx+xx} + \text{BNP} + \text{CL}$

$\text{Log. } \frac{x}{1+x^3} = \text{Fluent of } \frac{\dot{x}}{1+x^3}$. This in Fluxion
gives

$$\frac{Ba - As + Ax}{1 - 2sx + xx} \dot{x} + \frac{C\dot{x}}{1+x} = \frac{\dot{x}}{1+x^3}; \text{ put } Ba - As =$$

$$\text{then } \frac{1+Ax}{1-2sx+xx} + \frac{C}{1+x} = \frac{1}{1+x^3}; \text{ reduce}$$

them to a common Denominator, then

$$\frac{\begin{array}{r} + I + Ax + Ax^2 \\ + C + I \\ - 2sC + C \end{array}}{\begin{array}{r} 1 - 2sx + 1 \\ + 1 - 2sx^2 + x^3 \end{array}} = \frac{1}{1+x^3}$$

Here the Denominators being equal, we have $1 - 2sx + 1 = 0$, or $2s = 1$; and equating the Coefficients of
homologous Terms in the Numerator, $I + C = 0$,
 $A + I - 2sC = 0$ or $A + I - C = 0$, and $A + C = 0$.
From whence will be found $A = -\frac{1}{3}$, $C = \frac{1}{3}$, $I =$

$$B = \frac{1}{2a} = (\text{because } aa = \frac{3}{4},) \frac{2}{3}a. \text{ Whence}$$

Fluent is $-\frac{1}{3}L \times \text{Log. } \sqrt{1-2sx+xx} + \frac{2}{3}aNP$
 $\frac{1}{3}L \times \text{Log. } 1+x.$

This done, I put $x^\lambda = \frac{\beta z^n}{\alpha}$; and (by Prop. IX.)

transform the Fluxion $\frac{\dot{x}}{1+x^3}$ into this other $\frac{\eta \alpha^{\frac{2}{3}} \beta^{\frac{1}{3}}}{3} \times$

$\frac{z^{\frac{1}{3}\eta-1} \dot{z}}{\alpha + \beta z^n}$, whose Fluent therefore you have above.

In equating the Coefficients, if you had made $A+I-C=1$, and the other Equations $=0$, you would have got the Fluent of $\frac{x \dot{x}}{1+x^3}$.

After the same Manner may the 8th Form be investigated, remembring what was said of Sines and Logarithms, and observing that (if you put T, U, V, &c. for $1-x$, $\sqrt{1-2sx+xx}$, $\sqrt{1-2tx+xx}$), the first T into the Squares of all the rest, if λ is an odd Number; or the first and last into the Squares of the rest, if λ is an even Number, is always equal to $1-x^\lambda$: The Demonstration of which belongs not to this Place.

Here note that in these two Forms, you may if you please put P, Q, &c. for the Degrees of Arches

whose Sines are $\frac{a}{\sqrt{1-2sx+xx}}$, $\frac{b}{\sqrt{1-2tx+xx}}$ &c.

their Fluxions being the same as the other. But then

the Arch whose Sine is $\frac{ax}{\sqrt{1-2sx+xx}}$ be less

than a Quadrant, the Arch whose Sine is

$\frac{a}{\sqrt{1-2sx+xx}}$ will be greater than a Quadrant.

P R O P. XII.

To correct the Fluent of a given Fluxionary Equation or Proportion.

When the Fluent is obtained, by either of the foregoing Propositions, from any given fluxionary Equation or Proportion; it is only obtained in general. But since the Design in any particular Problem is to find the contemporary Fluents; such general Equation or Proportion therefore is for the most part imperfect till it be corrected by the following

R U L E.

1. Instead of each several variable Quantity (or Member) in the Fluent *substitute* such a *determined Value* thereof, as each of them is known to have in any one certain and *particular Point* or Place: Then *subtract* each *Side* of the resulting Equation from the *corresponding Side* of the *Fluent*; and the remaining Equation will be the *corrected Fluent*. And the same Rule obtains when the Fluent is expressed by a general Proportion.

2. Or any particular Part of the Fluent may be otherwise had thus without the foregoing Correction *Substitute* the Values of the variable Quantities for any particular *Time*, Place or Point; do the same for another given Time or Place; and the *Difference* of the resulting Equations, gives the corresponding Part of the *Fluent*.

3. Sometimes it may be sufficient to add some *given Quantity* on one Side of the Equation, which may afterwards be determin'd according to the Nature and Circumstances of the Question.

DEMON

DEMONSTRATION.

Let $X=Z$ be the Fluent obtained in general, from a given fluxionary Equation, $\dot{X}=\dot{Z}$. Now since X may not be equal to Z (by Cor. 2. Prop. II.), take d a given Quantity and let $X=Z+d$ be an Equation for the contemporary Fluents. Now at a certain Time when $X=b$, let $Z=c$; then you will have the particular Equation $b=c+d$ at that Time; this Equation taken from the former will leave this general Equation for the contemporary Fluents, $X-b=Z+d-c-d$, that is $X-b=Z-c$. Q. E. D.

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These Things are to be thus understood when the variable Quantity in the Fluent continually increases (or decreases). But in Case it increases and decreases by Turns, or passes through one or more *Maximums* or *Minimums*: Then the several Parts of the Fluent, between any given Point and each Maximum or Minimum must be *separately* found by distinct Operations, and each *corrected* by this Proposition, and then the several Parts *collected* into one *Sum*.

EXAMPLE I.

Let $a\dot{x}=2y\dot{y}$, and the Fluent is $ax=yy$, now when $y=0$, let $x=0$, and then $ax-0=y^2-0$, or $ax=yy$, which therefore needs no Correction.

Ex. 2.

Again let $a\dot{x}=2y\dot{y}$, and finding the Fluent $ax=yy$. Now when $y=0$, let $x=b$; and the Equation becomes $ab=0$, this subtracted from the other Equation, leaves $ax-ab=yy$, the contemporary Fluent required.

Ex. 3.

Let $b\dot{x} - x\dot{x} = y\dot{y}$, the Fluent is $b\dot{x} - \frac{x^2}{2} = \frac{y^2}{2}$
 but at a certain Point of Time $x = b$, and also $y =$
 then the Equation becomes $bb - \frac{1}{2}bb$ or $\frac{1}{2}bb = \frac{rr}{2}$
 therefore the correct Fluent is $b\dot{x} - \frac{1}{2}x\dot{x} - \frac{1}{2}bb =$
 $-\frac{1}{2}rr$. That is by Reduction $rr - bb + 2bx -$
 $= y\dot{y}$.

Ex. 4.

Suppose $2by\dot{y} = \frac{3cx^2\dot{x}}{z} - \frac{cx^3\dot{z}}{z^2}$, the Fluent
 $b\dot{y}y = \frac{cx^3}{z}$; but when $y = r$, the Quantity $\frac{cx^3}{z} =$
 Therefore the correct Fluent is $b \times \overline{y^2 - r^2} =$
 $-A$.

Ex. 5.

Let $\frac{\dot{y}^2}{2b} = \frac{\dot{x}\ddot{x}}{\sqrt{\dot{y}^2 + \dot{x}^2}}$, supposing $\dot{y}$ given; the
 Fluent is $\frac{y\dot{y}}{2b} = \sqrt{\dot{x}^2 + \dot{y}^2}$, but when $y = a$, $\dot{x} =$
 and $\dot{y} = \sqrt{\dot{y}^2 + \dot{x}^2}$, therefore the contemporaneous
 Fluent is $\frac{y\dot{y} - a\dot{y}}{2b} = \sqrt{\dot{x}^2 + \dot{y}^2} - \dot{y}$, or $y - a +$
 $\times \dot{y} = 2b\sqrt{\dot{x}^2 + \dot{y}^2}$.

Ex. 6.

Let $b\dot{y} = \frac{\dot{x}}{\sqrt{aa + xx}}$: By Form the 9th
 Fluent is $b\dot{y} = 2.3025 \text{ Log. } x + \sqrt{aa + xx}$; but
 when $y = a$, $x = 0$, then the Equation is $ba =$
 $2.3025 \text{ Log. } a$. Therefore the corrected Fluent
 $b\dot{y} - ba = 2.3025 \text{ Log. } x + \sqrt{aa + xx} - 2.3025$
 Log.

Log. a ; that is by $-ba = 2.302585$ Log.

$$\frac{x + \sqrt{aa + xx}}{a}$$

P R O P. XIII.

To investigate a Problem by the Method of Fluxions.

RULES.

1. Let all the Quantities be denoted by proper Symbols, as is explained in the Notation of Fluxions, and let some *one* of the variable Quantities (with which the others may always be compared) be supposed to increase uniformly: And this may be called the *Principal* variable Quantity. Then the given Equations, or such Equations as are deduced from the Conditions of the Problem, must be turn'd into Fluxions, second Fluxions, &c. by Prop. III. in order to get as many Equations among these Fluxions, as you have Occasion for.
2. But because sometimes some Doubt may arise about the Signs of the Fluxions: Observe that any fluxionary Equations, deduced from the Equations of Curves, or from any given Equations in the Problem, will contain the Fluxions with their proper Signs. But in any Proportions made between Fluxions at Moments, as in similar fluxionary Triangles and the like; then the Fluxions or Moments of Quantities that decrease must be actually made *negative*; and those that increase must be written *affirmative*: For, which is the same Thing, (since one Part of any Whole increases whilst the other Part of it decreases, therefore) instead of the negative Fluxion, you may take the proper Fluxion of the increasing Part of that Quantity.
3. Since

3. Since *Velocity* is always measured by the *Space* uniformly described thereby; so may the *Fluxions* be measured by the *Moments* uniformly generated by the *Fluxions*. Therefore the *Moments* (uniformly generated or, which amounts to the same Thing, considered as *arising* or *vanishing*) may be put for the *Fluxions*, and the Result will always be the very same in all Operations. And since in many Cases, especially in geometrical Problems, the reasoning and calculating with the *Moments* will be more easy and evident than with the *Fluxions*; the Equations that are gained thereby must at last be changed into fluxionary Equations, by substituting the *Fluxions* instead of the *Moments*, which must always be supposed to be taken in the *first Instant* of their Generation: Or, at least when the Operation is over, the *Moments* must be supposed to be *diminish'd ad infinitum*, that their *first Ratio* may be always obtained.

4. In the Resolution of any *Problem*, the Nature and Conditions of it are to be closely examin'd and strictly pursued according to all the known Methods of *Algebraic Reasoning*, by attentively considering the *Relations* of the Quantities, and their mutual *Proportions* and *Dependance* on one another; and forming your Process according to these their Properties, by duly comparing together the *Quantities*, their *Moments* or *Increments*, their *Fluxions* or *second Fluxions* &c. as the Case requires; till you get a competent Number of *Equations* or *general Proportions*. And then you must proceed to expunge such Quantities as are superfluous, till at last you get a fluxionary *Equation* or *Proportion* with the Quantities required. Then if there be Occasion,

5. Find the *Fluent* of the said Equation or general Proportion by Prop. X. or XI. and correct it by Prop. XII. And then you have a complete *Equation* or *general Proportion* containing the Quantities sought.

6. Be

6. But to obtain an *Equation* of the indetermin'd Quantities, by having the correct general *Proportion* before found, or by having only the fluxionary *Proportion*; you must assign to each indetermin'd Quantity in the said *Proportion*, (or in the *Fluxion* thereof) such a determined Value as it is known to have in any particular Case; and from thence you must draw an Analogy from the *Fluxion* alone (of the general *Proportion*), or from the *correct Fluent* alone, (or sometimes from both together.) From whence there will be had an *Equation* between the Quantities required; or at least between their *Fluxions*, whose *Fluent* must then be found and corrected. And note, these determin'd Values (of the Quantities) may be either express'd in Numbers or Symbols, as any one shall think proper.

Sometimes it may be sufficient to assume a given Quantity, by which multiplying one Side of the *Proportion* it will be turned into an *Equation*; and this given Quantity may afterwards be determined according to the Nature of the Question.

These are the general Rules, but after all, many Things must be left to the Sagacity and Invention of the Artist.

COROL. Hence every *Problem* belongs to *Fluxions*, in which the *Increments*, or the *Proportions* of the *Increments* or *Moments* of the several variable Quantities contained therein, can in all Cases be computed and expressed by *Equations*.

EXAMPLE I.

To find the Velocity wherewith the Ordinate *BM* of a Circle increases in every Point, whilst it moves uniformly along the Diameter *AD*. I.

Let $AD=2r$, $AB=x$, $BM=y$; and by the Nature of the Circle $2rx-xx=yy$; this Equation put into

F I G.
I.into Fluxions gives $2r\dot{x} - 2x\dot{x} = 2y\dot{y}$, or $\frac{r-x}{y} \dot{x} = \dot{y}$,general Equation for the Increase of y in all PointsTherefore in A where y is 0, and x is 0, $\frac{r-x}{y} \dot{x} =$ becomes $\frac{r\dot{x}}{0} = \dot{y}$, therefore $\dot{y}$ is infinite. If $CB = B$ or $r-x=y$, then $\dot{x}=\dot{y}$, and x and y increase equallyBut in C where $r-x=0$, then $\dot{y}=0$, therefore y doesnot increase at all. In b where $Cb=bm$, then $\dot{y}=-$ therefore y decreases as fast as x increases. Lastly D where $y=0$, and $x=2r$, $\dot{y} = \frac{-r\dot{x}}{0} = -$ Infinite

and there the Ordinate decreases infinitely: And in

all the intermediate Points it has all the intermediate

Degrees of Increase or Decrease.

E x. 2.

*To find the Space a descending Body will describe in a
Time by the uniform Force of Gravity.*Let $z =$ Space, $x =$ Time, $v =$ Velocity, acquiredin that Time. By the Principles of Mechanics $z \propto$ $v\dot{x} \propto x\dot{x}$, and therefore $\dot{z} \propto v\dot{x} \propto x\dot{x}$; that is the

Fluxion of the Space is every where as the Velocity

into the Fluxion of the Time; that is, (because the

Velocity is as the Time) as the Time into the Fluxion

of the Time.

Now if only the Ratio of the Space and Time be

required, it will be sufficient to take the Fluent; and

then $z \propto \frac{x^2}{2}$, or $z \propto x^2$, that is, the Space is always as

the Square of the Time.

But if the absolute Quantity of the Space is sought, we must reduce the general Proportion or its Fluxion to an Analogy from some particular known Case

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Thus it is known by Experiment that in the Time t F I G. or 1 Second, a Body would acquire such a Velocity as to move through s or $32\frac{1}{2}$ Feet uniformly in that Time, or to have descended through $\frac{1}{2}s$ or $16\frac{1}{2}$ Feet in that Time. Whence

$t:s::\dot{x}:\frac{s\dot{x}}{t}$ = Fluxion of the Space z when $x=t$; wherefore from the general Analogy ($\dot{z}\propto x\dot{x}$) we have $\dot{x}:t\dot{x}::\dot{z}:x\dot{x}$, and $\dot{z}=\frac{sxx}{tt}$, and finding the Fluent, $z=\frac{sxx^2}{2tt}$, which needs no Correction (because when $z=0$, $x=0$).

Or thus from the Fluxion and Fluent, $\frac{s\dot{x}}{t}:t\dot{x}::z:$ $\frac{1}{2}$, whence $z=\frac{sxx}{2tt}$.

Or lastly thus, since $\frac{1}{2}s=z$, when $t=x$, therefore from the general Analogy ($z\propto xx$) it will be $\frac{1}{2}s:tt::z:xx$; whence $z=\frac{sxx}{2tt}$, and thus the same Equation is obtained any of these Ways.

Ex. 3.

If a Body is projected upwards with a given Velocity a , to find how far it will ascend in any Time x .

Let z = Space, v = its Velocity; then by Mechanics $\dot{z}\propto v\dot{x}$. Now since the first Velocity is given, therefore the Space s , which would be uniformly described in the Time t , of its whole Ascent,

will be given; wherefore $t:s::\dot{x}:\frac{s\dot{x}}{t}$ = Moment of the Space $\dot{z}$ at the first Instant, that is when $a=v$.
T There-

F I G. Therefore from the general Proportion ($\dot{z} \propto v\dot{x}$),
 1.

$$: a\dot{x} :: \dot{z} : v\dot{x} :: \dot{z} : v\dot{x}; \text{ whence } \dot{z} = \frac{sv\dot{x}}{ta}; \text{ but}$$

the Velocity the Body loses is as the Time, therefore $t : a :: x : \frac{ax}{t} = \text{Velocity lost, whence}$

$$v = a - \frac{ax}{t}, \text{ therefore } \dot{z} = \frac{s\dot{x}}{t} - \frac{sx\dot{x}}{tt}; \text{ and}$$

the Fluent $z = \frac{sx}{t} - \frac{sx^2}{2tt}$, which needs no Correction.

Hence if x be greater than $2t$, the Body will have descended again below the Point it was projected from.

Otherwise thus.

Let $z = \text{Space}$, $v = \text{Velocity}$, $s = \text{Velocity generated by Gravity in the Time } t$; then by *Mechanics*

$\dot{z} \propto v\dot{x}$. Now since the first Velocity is given; therefore the Space d , which would thereby be uniformly described in a given Time t , will be given; where

fore $t : d :: \dot{x} : \frac{d\dot{x}}{t} = \text{Moment of the Space at the first Instant, that is when } a = v$. Therefore

from the general Proportion ($\dot{z} \propto v\dot{x}$) $\frac{d\dot{x}}{t} : a\dot{x} ::$

$$: v\dot{x} :: \dot{z} : v\dot{x}, \text{ whence } \dot{z} = \frac{dv\dot{x}}{ta}; \text{ but the Velocity the Body loses is as the Time; therefore}$$

$t : s :: x : \frac{sx}{t} = \text{Vel. lost; whence } v = a - \frac{sx}{t}$

therefore $\dot{z} = \frac{d\dot{x}}{t} - \frac{dsx\dot{x}}{att}$; and the Fluent $z =$

$\frac{dx}{t} = \frac{dsxx}{2att}$; but $t : d :: 1 : a = \frac{d}{t}$; there- F I G.

fore $z = ax - \frac{sxx}{2t}$.

Hence if $2ta$ be greater than sx , the Body will have descended again below the Point it was projected from.

Ex. 4.

To find the Time wherein a given Cylinder full of Water will empty itself by a Hole at the Bottom.

Let $AC=b$, $CE=x$, $AE=b-x$, t = Time of running out with the first Velocity, z = Time sought. 2.

Now the Moment of the Quantity running out $\propto \dot{z}$ x Velocity; but the Velocity is as $\sqrt{CE}$, and the Moment of the Quantity is as the Moment of AE ,

or $-\dot{x}$, whence $-\dot{x} \propto \dot{z}\sqrt{x}$, therefore $\dot{z} \propto \frac{-\dot{x}}{\sqrt{x}}$;

But $b : t :: -\dot{x} : \frac{-t\dot{x}}{b}$ = Moment of the Time at the first Instant. Therefore (from the general

Proportion $\dot{z} \propto \frac{-\dot{x}}{\sqrt{x}}$), $\frac{-t\dot{x}}{b} : \frac{-\dot{x}}{\sqrt{b}} :: \dot{z} : \frac{-\dot{x}}{\sqrt{x}}$;

$\dot{z} : \frac{-\dot{x}}{\sqrt{x}}$, whence $\dot{z} = \frac{-t\dot{x}}{\sqrt{bx}}$, and the Fluent is

$z = \frac{-2tx^{\frac{1}{2}}}{\sqrt{b}}$; but in the Point A , $z=0$, and

$z=b$, therefore the Fluent corrected (by Prop. XII.)

$z = 2t - \frac{2tx^{\frac{1}{2}}}{\sqrt{b}}$; and when $x=0$, then the whole

Time $z = 2t$.

Ex. 5.

FIG. To find the Time wherein a given Frustum of a Cone
 3. full of Water will empty itself by a Hole in the Bottom.

Let the Cone be compleated, and put $VG=p$ the Altitude; $TG=b$ the Height of the Frustum, $TG=x$, Circle $CD=d$, $VT=b$, t =Time of running out (of a Cylinder whose Base is d and Height b) with the first or greatest Velocity, z =Time sought. Proceeding here as in the last Example, you'll find

$\dot{z}\sqrt{x} \propto$ Moment of the Quantity $\propto -\dot{x} \times$ Circle

$EF \propto \frac{-\dot{d}x}{pp} \times \overline{b+x^2}$. Therefore $\dot{z} \propto \frac{-\dot{d}x}{pp\sqrt{x}}$

$\times \overline{b+x^2}$; then $\frac{-tx}{b} : \frac{-\dot{d}x}{\sqrt{b}} :: \dot{z} : \frac{-\dot{d}x}{pp\sqrt{x}} \times \overline{b+x^2}$

$z : \frac{-\dot{d}x}{pp\sqrt{x}} \times \overline{b+x^2}$, from the general Analogy;

therefore $\dot{z} = \frac{-tx}{pp\sqrt{bx}} \times \overline{b+x^2}$. And the

Fluent is $z = \frac{-t}{pp\sqrt{b}} \times : 2bbx^{\frac{3}{2}} + \frac{4}{3}bx^{\frac{5}{2}} + \frac{2}{5}x^{\frac{7}{2}}$

and when corrected, the whole Time $z = t \times \frac{2bb + \frac{4}{3}bb + \frac{2}{5}bb}{pp}$.

And if the Frustum were inverted, the Time would be found to be $t \times \frac{2pp - \frac{4}{3}pb + \frac{2}{5}bb}{bb}$. Where d =Circle AB , x = GS .

SCHOLIUM.

If EF or y be the double Ordinate in any Curve CA ; the Time of running out might have been found the same Way, only by substituting the Value of E

or y had from the Nature of the Curve, into the F I G.

Equation $\dot{z} = \frac{-ty^2 \dot{x}}{d\sqrt{bx}}$, and then finding the Fluent,

where d = Square of the Diameter of any Cylinder,
and b = Height, t = Time of the Cylinders running
out with that first Velocity.

Ex. 6.

Let ACE be (the Section of) a Wall supporting a Fluid
behind it, and joining to the perpendicular Side AC ;
to find the Curve ADE terminating the other Side of
the Wall, so that its Strength may be every where as
the Pressure it sustains.

Let $AC = b$, $AB = x$, $BD = y$. The Effect which
any Number of Particles of the Fluid pressing at B
have to break the Wall at C , is as $CB \times$ Number of

particles $\times$ their Force, that is $\propto \overline{b-x} \times \dot{x} \times x$, (be-

cause the Number is as $\dot{x}$, and the Pressure or Force
as x). And the Sum of all the Forces acting on AB

to break it at C is as the Sum of all the $\overline{b-x} \times x \dot{x}$
that is as the Fluent of $bxx - x^2 \dot{x}$, and therefore as

$\frac{bx^2}{2} - \frac{x^3}{3}$, and when $x = b$, the whole Pressure on AC

to break it at C will be $\frac{1}{6}b^3$; therefore the Effects of
the Pressure at B and C will be as AB^3 and AC^3 . But

the Strength of the Wall in B and C is supposed to
be as these Forces, and by Mechanics 'tis known to

be as BD^2 and CE^2 ; Therefore $AB^3 : AC^3 :: BD^2 :$
 CE^2 , that is $y^2 \propto x^3$; And the Curve ADE is a Semi-

cubical Parabola whose Vertex is A , and therefore
convex towards AC .

Ex.

Ex. 7.

- FIG. Suppose a Wind to blow against the perpendicular
 4. *AC* of the Wall *ACE*; to find the Curve *AD* bounding the other Side, so as the Strength of it every where as the Force it sustains.

Let $AB=x$, $BD=y$, $AC=b$. The Force of all the Particles at *B* to break the Wall at *C* is as $CB \times$ Number of Particles $\propto \overline{b-x} \times \dot{x}$ or $\propto \overline{b-x} \times \dot{x}$; and therefore the whole Force of all the Particles on *AB* to break the Wall at *C* is $\propto$ the Fluent of $b\dot{x} - x\dot{x}$ $\propto bx - \frac{x^2}{2}$; therefore the Force to break it at *C* by all the Particles on *AC* is $\frac{1}{2}bb$ or as bb , and this may be as the Strength of the Wall or as CE^2 ; consequently $x^2 \propto y^2$ and $x \propto y$, therefore *ADE* is a right Line.

Ex. 8.

5. Let *ABC* be a heavy Body, *BC* a Spring fixt to the Block *D*: Let *AB* be close thrust up to *C*, that the Spring may be close bent, and fixt thus to the Stock by a Pin. To find with what Velocity the Body may be projected by the Force of the Spring when the Pin is suddenly pluck'd out.

Let $BC=b$ the Length of the Spring in its natural Position, $CN=x$, v = Velocity of the Point *B* when it arrives at *N*, w = Weight of the Body *AB*, t = Time of describing *CN*. By Mechanics or the Law

of Motion $\dot{v} \propto \frac{\text{force} \times t}{\text{Body}} \propto$ (by the Nature of

Spring) $\frac{b-x}{w} t$: Likewise by the Laws of Motion

$\dot{t} \propto \frac{\dot{x}}{v}$; therefore $\dot{v} \propto \frac{b-x}{w} \times \frac{\dot{x}}{v}$ or $v\dot{v} \propto \frac{b\dot{x} - x\dot{x}}{w}$, and finding

finding the Fluent, $v^2 \propto \frac{2bx - xx}{w}$, and when $x = b$, F I G.

$\propto \frac{bb}{w}$; that is the Square of the Velocity is reciprocally as the Weight of the Body: Consequently, the Body is projected horizontally, the Square of the Distance it is projected to will also be reciprocally as the Body.

Universally, since $b - x$ represents the Force of the Spring at N ; and b the Force of it at C ; instead of b , we put f = absolute Force of the Spring at C , then will $\frac{bf}{w}$ be as the Square of the Velocity for all Springs.

Ex. 9.

Let BC be the Quadrant of a Circle, A the Center, RS parallel to BA . To find the Nature of the Curve DFQ that constantly bisects the Angle made by RF and the Arch FC . 6.

Describe the Circle ne infinitely near FC , and draw on parallel to BA , and no parallel to CA ; then since the Angle $Fon = nFp$, and Side Fn common, and the Angles o, p right, therefore $on = np$.

Let $AS = x$, $SF = y$, AB or $Ap = v$, $AD = b$,

and since $on = np$, that is $\dot{x} = \dot{v}$, therefore $\dot{x} = \dot{v}$, and the Fluent $x = v$, and corrected $x = v - b$, or

$x = v$. But by Prop. 47. Eu. I. $\overline{b+x}^2 (= v^2)$ $yy + xx$, that is $bb + 2bx = yy$; whence the Curve FQ is a Parabola, whose Latus rectum is $2b$, and focus the Point A .

Ex.

Ex. 10.

FIG. To find the Time of the Vibration of a Pendulum in
 7. extremely small Arch of a Circle; or in any other
 Arch.

Let the Length of the Pendulum $CB=r$, $CA=c$, $BF=x$, Arch $BE=v$, $Ee=\dot{v}$, $Ff=\dot{x}$, the
 $BD = \frac{cc}{2r}$, $BG = \frac{cx}{2r}$, $DG = \frac{c}{2r} \times c-x$, EG

$\sqrt{cx - \frac{ccxx}{4rr}}$. Also let $t =$ Time of a Body
 descending or ascending thro' the Cord AB , z
 Time of descending or ascending thro' the Arch
 BE , then $\frac{1}{2}t =$ Time of describing AB with the
 Velocity in B ; by the Nature of the Circle $\dot{x}$

$\frac{cx}{2\sqrt{cx - \frac{ccxx}{4rr}}}$. The Times of describing

Spaces uniformly are as the Spaces directly and the
 Velocities reciprocally; but since the Pendulum falls
 from A and is supposed to describe the Arch AB
 descending, or BA in ascending, therefore the Veloc-
 ities in B and E are as $\sqrt{DB}$ and $\sqrt{DG}$, or

$\sqrt{AB}$ and $\sqrt{AF}$; therefore $\frac{1}{2}t : \frac{c}{\sqrt{c}} :: \dot{z} : \dot{x}$

$:: \dot{z} : \frac{\dot{v}}{\sqrt{c-x}}$; whence $\dot{z} = \frac{t\dot{v}}{2\sqrt{cc-cx}}$

$\frac{tx^{-\frac{1}{2}}\dot{x}}{2\sqrt{4crr - \frac{4rr}{cc}x + cxx}} =$ (rejecting cx

as extremely small) $\frac{tx^{-\frac{1}{2}}\dot{x}}{4\sqrt{c-x}}$: And the Fluxion

(by Form 10th) is $z = \frac{t}{2r} \times \text{Arch of this Circle F I G.}$

7.

whose Sine is $r\sqrt{\frac{x}{c}}$, and when $x=c$, then $z =$

$\frac{t}{4} \times \text{Quadrant BH} = \frac{t}{4} \times 3.1416$: And $2z$ or

the Time of an entire Vibration is $= \frac{1}{2}t \times 3.1416$;

Or, which is the same Thing, $2z = \frac{3.1416}{2} \times \text{Time}$

of descending through twice the Length of the Pendulum. Or putting $s = 16\frac{1}{2}$ Feet, $r =$ Feet in

the Pendulum, then $2z = \frac{3.1416}{2} \times \sqrt{\frac{2\eta}{s}}$, in Se-

conds.

And to find the Time of Vibration in any Arch

AB; we have $z = \frac{trx^{-\frac{1}{2}}\dot{x}}{2\sqrt{4crr-4r^2x-cx+cxx}} =$

$\frac{tx^{-\frac{1}{2}}\dot{x}}{\sqrt{c-x}\sqrt{1-\frac{cx}{4rr}}} = \frac{tx^{-\frac{1}{2}}\dot{x}}{4\sqrt{c-x}} \times 1 - \frac{cx}{4rr}^{-\frac{1}{2}}$

$= \frac{tx^{-\frac{1}{2}}\dot{x}}{4\sqrt{c-x}} \times (1-dx)^{-\frac{1}{2}}$ (putting $d = \frac{c}{4rr}$) =

$\frac{tx^{-\frac{1}{2}}\dot{x}}{\sqrt{c-x}} \times : 1 + \frac{1}{2}dx + \frac{1.3}{2.4}d^2x^2 + \frac{1.3.5}{2.4.6}d^3x^3 \&c.$

whose Fluent, by Form 17th, is $\phi \times : 1 + \frac{cd}{2^2} +$

$\frac{3^2}{2^2.4^2}c^2d^2 + \frac{3^2.5^2}{2^2.4^2.6^2}c^3d^3 \&c. = \phi \times : 1 + \frac{cc}{2^2.4rr}A$

$+ \frac{3^2cc}{4^2.4rr}B + \frac{5^2cc}{6^2.4rr}C.$ Where $\phi = F: \frac{tx^{-\frac{1}{2}}\dot{x}}{4\sqrt{c-x}}$

$= \frac{3.1416t}{4}$, the Time in a very small Arch.

U

There-

F I G. Therefore the Time of Vibration in a very small Arch, is to the Time in the Arch, the Chord

whose Half is c , as 1 to $1 + \frac{cc}{2^2.4rr}A + \frac{3^2cc}{4^2.4rr} + \frac{5^2cc}{6^2.4rr}C + \frac{7^2cc}{8^2.4rr}D + \mathcal{E}c.$

COR. Hence if the *Pendulum* r measures Time vibrating in a very small Arch; and if c be the Chord of the Arch a ; then the Seconds lost in 24 Hours vibrating in the Arch $2a$, will be nearly $\frac{86400 \times c}{10rr}$

or $\frac{5400cc}{rr}$, and the Minutes lost $\frac{90cc}{rr}$.

Or if Time be measured by vibrating in the Arch $2a$, then the Seconds lost by vibrating in the Arch a (C being the Chord of A) will be $\frac{5400}{rr} \times CC$ nearly.

EX. II.

8. To find the Meridional Parts for any Latitude.

Let Radius $CA = r$, the given Arch of Latitude $AB = v$, Sine $DB = y$, Meridional Parts of $AB = z$. By Construction of *Mercator's Chart*, as Cosine

the Latitude $(\sqrt{rr - yy}) : \text{Radius } (r) :: (\dot{v} : \dot{z})$

$\dot{v} : \dot{z} = \frac{r\dot{v}}{\sqrt{rr - yy}}$, but by the Nature of the

Circle $\dot{v} = \frac{ry}{\sqrt{rr - yy}}$; whence $\dot{z} = \frac{ry}{rr - yy}$

whence, by Form the 6th, $z = \frac{2.30258r}{2} \times \text{Log.}$

$\frac{r+y}{r-y} = 2.30258r \times \text{Log.} \sqrt{\frac{r+y}{r-y}}$. But in the

Triang

Triangle EBF , as $EB (\sqrt{rr-yy}) : \text{Rad. } (r) :: EF \text{ I G.}$

$$(r+y) : \text{Tangent of the Angle } B = \frac{r \times r+y}{\sqrt{rr-yy}}$$

$$= r \sqrt{\frac{r+y}{r-y}} = \text{Cotangent of half the Complement}$$

of the Latitude AB , whence $z = 2.3025r \times \text{Log.}$
Cotangent of half the Complement of the Latitude.

And by Correction $z = 2.302585r \times \text{Log. Cot. of half}$
the Co. Lat. — $2.30258r \times \text{Log. Rad.}$ and this is the

Arch of the Nautical Meridian. But since the

Meridional Parts in the Tables are expressed in
Minutes, therefore $z = \frac{2.302585 \times 180 \times 60}{3.14159}$

$$\text{Log. Cot. } \frac{1}{2} \text{ Co. Lat. — Log. Rad.} = 7915.705 \times \frac{\text{Radius}}{\text{Cot. } \frac{1}{2} \text{ Co. Lat.}}$$

$$\text{Log. } \frac{\text{Rad.}}{\text{Cot. } \frac{1}{2} \text{ Co. Lat.}} = 7915.705 \times \text{Log. Tan. } \frac{1}{2} \text{ Co. Lat.}$$

COR. 1. Hence the Meridional Parts for the
Difference of Latitude of two Places is $= 7915.7 \times$
Difference of the Log. Tangents of half their Comple-
ments of Latitude. *as*

COR. 2. Since, Radius : Meridional Difference
Latitude :: so Tangent of the Course : to
the Difference of Longitude; and $\frac{\text{Rad.} = 1}{7915.705} =$

000126331, therefore

As ,000126331 :

To Tangent of the Course ::

So the Diff. of the Log. Tangents of half the Comp.
the Lat. :

To the Difference of Longitude.

Otherwise thus.

Let $t = \text{Tang. of the Latitude } AB$, $DC = x$;
then by the Nature of the Circle $yy = rr - xx$ and

U 2

yy

FIG. 8. $y\dot{y} = -x\dot{x}$, and $\dot{y} = \frac{-x\dot{x}}{y}$. But by Trigonometry

$$\dot{y} \left(\frac{-x\dot{x}}{y} \right) : -\dot{x} :: \text{Rad. (1)} : t = \frac{y}{x}, \text{ and}$$

$$= tx. \text{ Likewise } x : r :: \dot{y} : \dot{v} = \frac{r\dot{y}}{x}. \text{ And by}$$

the Nature of Mercator's Projection, $x : r ::$

$$\left(\frac{r\dot{y}}{x} \right) : \dot{z} = \frac{rr\dot{y}}{xx} = \frac{-rr\dot{x}}{xy}. \text{ But } t^2x^2 = yy =$$

$$rr - xx, \text{ whence } x = \frac{r}{\sqrt{1+tt}}, \text{ and } y =$$

$$\frac{rt}{\sqrt{1+tt}}, \text{ and } \dot{x} = \frac{-rt\dot{t}}{(1+tt)^{\frac{3}{2}}}, \text{ and } xy =$$

$$\frac{rrt}{1+tt}; \text{ therefore } \dot{z} = \frac{-rrx \cdot -rt\dot{t}}{(1+tt)^{\frac{3}{2}}} \times \frac{1+tt}{rrt} =$$

$$\frac{rt\dot{t}}{\sqrt{1+tt}}. \text{ Therefore, by Form the 9th, } z =$$

$2.30258r \times \text{Log. } t + \sqrt{1+tt}$: But if t is the Tangent, $\sqrt{1+tt}$ is the Secant; and by the Elements of Trigonometry, Secant + Tangent = Cotan. of half the Complement of AB . Therefore $z = 2.30258r \times \text{Log. Cotangent of half the Complement of the Latitude } AB$.

To find the Meridional Parts for the oblate Spheroid.

Let the Semitransverse $CA = r$, Semiconjugate $CG = c$, Arch $AB = v$, Ordinate $DB = y$, $CD =$ Meridional Parts of AB , put $= Z$; and $t =$ Tan. of the Lat. at B , as before.

Then by the Nature of the Figure $ccrr - ccxx = rryy$, and $-ccx\dot{x} = rry\dot{y}$. But by Trigonometry

$$\dot{y} : -\dot{x} \left(\frac{rry\dot{y}}{ccx} \right) :: \text{Rad. (1)} : t = \frac{rry}{ccx}. \text{ And}$$

Mercator's Projection, $x : r :: \dot{v} : \dot{z} = \frac{r\dot{v}}{x}$.

And since $ccxt = rry$, we shall have $x = \frac{rr}{\sqrt{rr + cctt}}$,

$y = \frac{cct}{\sqrt{rr + cctt}}$, and $\dot{y} = \frac{rrcct\dot{t}}{\sqrt{rr + cctt}^{\frac{3}{2}}}$, $\dot{x} =$

$\frac{-rrcct\dot{t}}{\sqrt{rr + cctt}^{\frac{3}{2}}}$; but $\dot{v} = \sqrt{\dot{x}^2 + \dot{y}^2} =$

$\frac{\sqrt{r^4c^4t^2\dot{t}^2 + r^4c^4t^2}}{(rr + cctt)^{\frac{3}{2}}} = \frac{rrcct\dot{t}\sqrt{1 + tt}}{rr + cctt\sqrt{rr + cctt}}$; there-

fore $\dot{z} \left(\frac{r\dot{v}}{x} \right) = \frac{r^3cct\dot{t}\sqrt{1 + tt} \times \sqrt{rr + cctt}}{rr + cctt\sqrt{rr + cctt} \times rr} =$

$\frac{rct\dot{t}\sqrt{1 + tt}}{rr + cctt}$; or $\dot{z} = \frac{rct\dot{t} \times \sqrt{1 + tt}}{rr + cctt\sqrt{1 + tt}} = \frac{r\dot{t}}{\sqrt{1 + tt}}$

$\frac{rct\dot{t} - r^3\dot{t}}{rr + cctt\sqrt{1 + tt}}$. Put $rr - cc = ff$, $\sigma =$

of sec. Latitude, then $\dot{z} = \frac{r\dot{t}}{\sqrt{1 + tt}} - \frac{rff\dot{t}}{rr + cctt\sqrt{1 + tt}}$

$\dot{z} - rff \times \frac{t^{-3}\dot{t}}{cc + rrt^{-2}\sqrt{1 + t^{-2}}} =$ (by Form

e 23d) $\dot{z} - rff \times \frac{cc - rr + rrv}{cc - rr + rrv}^{-1} \times \frac{v^{-\frac{1}{2}}\dot{v}}{-2} =$

$-\frac{rff}{2} \times \frac{v^{-\frac{1}{2}}\dot{v}}{rr - cc - rrv}$; whose Fluent, by

Form the 6th, is $Z = z - \frac{rffL}{2r\sqrt{rr - cc}} \times \text{Log:}$

$r\sqrt{v}$

FIG. 8. $\frac{r\sqrt{v+\sqrt{rr-cc}}}{r\sqrt{v-\sqrt{rr-cc}}} = z - \frac{1}{2}fL \times \text{Log.} \frac{r\sqrt{1+\frac{1}{tt}}}{r\sqrt{1+\frac{1}{tt}}}$

$= z - \frac{1}{2}fL \times \text{Log.} \frac{r\sigma + f}{r\sigma - f}$, because by Trigonometry

$\sqrt{1 + \frac{1}{tt}}$ or $\frac{\sqrt{1+tt}}{t} = \sigma$. In Minutes $Z = z -$

$7915.7 \frac{f}{r} \times \text{Log.} \sqrt{\frac{\sigma + \frac{f}{r}}{\sigma - \frac{f}{r}}}$. Or putting $\frac{f}{r}$

$\frac{\sqrt{rr-cc}}{r} = d$, then will $Z = z - \frac{7915.7d}{2} \times \text{Log.}$

$\frac{\sigma+d}{\sigma-d}$. Where $\sigma =$ natural Coscant of the Latitude

tude to the Radius 1, and $z =$ the Meridional Part for the Sphere.

Or thus.

Since $\frac{1}{\sigma} = y$, therefore $\frac{\sigma+d}{\sigma-d} = \frac{1+dy}{1-dy}$, therefore

$Z = z - 7915.7d \times \text{Log.} \sqrt{\frac{1+dy}{1-dy}}$. But by Trigonometry,

$\sqrt{\frac{1+dy}{1-dy}} = \text{Cotan.} \frac{1}{2} \text{ Arch whose natural}$

Cosine is dy . Therefore, let $y = \text{nat. S. Lat. } A =$ Arch whose nat. Cosine is dy , to the Rad. 1; and $B = \text{Log. Cotan.} \frac{1}{2} A - 10$, as had in the Tables then $Z = z - 7915.7dB$.

In the spheroidal Figure of the Earth, we shall have $d = .093$, or thereabouts, according to Sir I. Newton; or $d = .148$, according to Maupertuis.

Ex. 12.

Find the Nature of the Curve which a heavy flexible Line will form itself into by its Gravity. FIG. 9.

Let the Line be suspended on the two fixt Points P , and dispose itself into the Curve DAP ; A its vertex, AQ its Axis, and BC an Ordinate.

1. The Part of the Curve ABD is kept in its Position by a certain Force at A acting in Direction AZ parallel to the Horizon; For if the Line be cut thro' A it will reduce itself to a perpendicular Position: and this Force acting at A is always the same whatever Length the Curve be of; For if the Line be cut thro' at B , and then the Point B fastned to the Plane; it is evident the Force at A is neither greater or less; for the Resistance of the point B does the same as the Tension of the Line in B did before; and the Force in A or the Tension of the Line in A must remain the same.

2. Let a = the given Part of the Line whose Weight is equal to the Tension of the Line in A , and $AC=x$, $BC=y$, $AB=z$, draw the Tangent BS , and $BR=BA$, perpendicular to the Horizon, and RS parallel to it. The Line BA is sustained by three Forces, for its Gravity acts in Direction BR , it is drawn at A in Direction AZ by the Force a , and it is sustained in B by the Tension of the Line in Direction SB ; and these three Forces being as BR , RS , and BS , and $BR=z$ by Construction, therefore the Force $a=RS$: Whence by similar Triangles $BR(z)$:

$RS(a) :: bo(\dot{x}) : B o(\dot{y}) :: \dot{x} : \dot{y}$, and $a\dot{x} = z\dot{y}$, which is one Property of the Curve.

3. Take $Br=Bb$ the Increment of the Curve, draw rn parallel and Bn perpendicular to BS , then $nB=Bo$, and $rn=bo$. Since BR is the perpendicular

F I G. dicular Force or Weight of the Line at B, Br

9. z is the Increment thereof, and therefore $\dot{x}$ is the Increment of the Tension of the Line in B, and $\dot{x}$ the Fluxion of the Tension, and therefore the Fluxion $\dot{x}$ = Tension or Force acting in Direction nr ; But in A where $x=0$, this Tension $=a$, therefore by Corollary, the whole Tension drawing in Direction of the Curve is $a+x$; and this is the Force BS, as was shewn before: Therefore again by similar Triangles $a+x (BS) : z (BR) :: \dot{z} : \dot{x} :: \dot{z} : \dot{x}$, whence $a\dot{x} = z\dot{z}$, and the Fluent $2ax + xx = zz$, which is another Property of the Curve.

4. If the Point B be so taken that the Angle R or SBC be half a right Angle, then will AB or z $=a$; for then $\dot{x}=\dot{y}$, and $z = \left(\frac{a\dot{x}}{\dot{y}} = \right) a$.

5. Since $\dot{y} = \frac{a\dot{x}}{z} = \frac{a\dot{x}}{\sqrt{2ax+xx}} = \frac{a\dot{z}}{\sqrt{aa+zz}}$ $\frac{a\dot{z}}{a+x}$; therefore by Form 9th, $y = 2.30258a \times \text{Log.}$

$$\frac{a+x+\sqrt{2ax+xx}}{a} = 2.30258a \times \text{Log.} \frac{z+\sqrt{aa+zz}}{a}$$

whence the Curve may be easily constructed.

Ex. 13.

10. To find the Nature of the Curve BM in which a Body moving (after its Fall thro' AB), it shall descend equal Spaces in equal Times.

Let $AB=a$, $BP=x$, $PM=y$, now since the Velocities of Bodies are as the Spaces described in equal Times, and the Squares of the Velocities are as the Heights fallen from; therefore $a : a+x :: (\text{Square of the Velocity in the Axis at P} : \text{Square of the Velocity in the Axis at B})$

velocity in the Curve at $M :: Pp^2 : Mm^2 :: \dot{x}^2 : \dot{x}^2 + \dot{y}^2 ::$) F I G.
 $\dot{x}^2 : \dot{x}^2 + \dot{y}^2$, and by Division $a : x :: \dot{x}^2 : \dot{y}^2$; there- 10.
 fore $x\dot{x}^2 = a\dot{y}^2$, or $x^{\frac{1}{2}}\dot{x} = a^{\frac{1}{2}}\dot{y}$, and finding the Fluent
 $x^{\frac{3}{2}} = a^{\frac{1}{2}}y$, or $ay^2 = \frac{4}{3}x^3$: Therefore the Curve is a
 semi-cubical Parabola convex towards BP .

Ex. 14.

a Body be projected from any Point A parallel to the
 horizontal Plane BC , and be urged towards that
 Plane with a Force which is as any Power of its
 Height above the Plane; to find the Nature of the
 Curve it will describe.

II.

Let $AB=r$, $AD=x$, $DP=y$, v = Velocity ac-
 quired in falling through AD , t = Time of falling,
 f = Force in D or P , which suppose to be as BD^n .

Now from the Laws of Motion $\dot{v} \propto ft$, but $t \propto$

and $f \propto r - x^n$, therefore $\dot{v} \propto \frac{r - x^n}{v}$, and

$\frac{r - x^n}{v} \dot{x}$ or $v\dot{v} \propto r - x^n$; and finding the Flu-

$\frac{v\dot{v}}{2} \propto \frac{r - x^n}{n+1}$, but in A , $x=0$, therefore

Fluent corrected is $v\dot{v} \propto \frac{r^{n+1} - (r - x^n)^{n+1}}{n+1}$.

the Fluxion of the Axis is as the Velocity of a
 descending Body, and the Fluxion of the Ordinate
 as the Fluxion of the Time, or as a given Quan-

b; therefore $\dot{x} : \dot{y} :: (v =) \sqrt{\frac{r^{n+1} - (r - x^n)^{n+1}}{n+1}}$

X

b,

F I G. : b , whence $\dot{y} = \frac{b\dot{x}}{\sqrt{\frac{r^n+1}{n+1} - \frac{r-x^{n+1}}{n+1}}}$, or $y =$

$\frac{b\dot{x}}{\sqrt{\frac{r^n+1}{n+1} - \frac{r-x^{n+1}}{n+1}}}$, and the Fluent will give
the Nature of the Curve.

12. CASE 1. Let $n=1$, then $\dot{y} = \frac{b\dot{x}}{\sqrt{2ax-xx}}$; describe
the Quadrant AEF ; then by the Nature of the Circle, Arch $AE =$ Fluent of $\frac{r\dot{x}}{\sqrt{2rx-xx}}$, therefore

take y or $DP = \frac{b}{r} \times$ Arch AE , and P will be
the Curve required.

CASE 2. Let $n=0$, then $\dot{y} = \frac{b\dot{x}}{\sqrt{x}}$, and $y = 2b\sqrt{x}$
or $4bbx=yy$; therefore the Curve will be a Parabola
as is well known.

13. CASE 3. Suppose $n=-1$. Here we must have
Recourse to the original Process itself, and there
shall have $v\dot{v} \propto \frac{\dot{x}}{r-x}$; describe the Hyperbola AEH
to the Asymptotes AB, BC ; then the Area $ADEH$
Fluent of $\frac{\dot{x}}{r-x}$; therefore $v \propto \sqrt{ADEH}$, and $y =$

$\frac{b\dot{x}}{\sqrt{\text{Area } ADEH}}$ for the Nature of the Curve AEH

14. CASE 4. Let $n=-2$, then $\dot{y} = b\dot{x}\sqrt{\frac{rr-x}{x}}$
Let AE be a Cycloid, $AG = v$, $DE = u$, AO
being the generating Circle; then $u = v + \sqrt{rx}$

and $\dot{u} = \dot{v} + \frac{r-2x}{2\sqrt{rx-xx}} \dot{x} =$ (because $\dot{v} = F I G.$

14.

$\frac{r\dot{x}}{\sqrt{rx-xx}}, \dot{x}\sqrt{\frac{r-x}{x}}.$ But $\dot{y} = br^{\frac{1}{2}} \times \dot{x}\sqrt{\frac{r-x}{x}},$

whence $y = bu\sqrt{r}$, therefore take $DP = b\sqrt{r} \times DE$, and P will be in the Curve.

CASE 5. Let $n = -3$; then $\dot{y} = \frac{br \times r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}},$

whence $y = br\sqrt{2rx - xx}$, and the Curve is an Ellipsis whose Semi-axis is AB .

Ex. 15.

To know whether a given Curve Line be concave or convex towards its Axis, in any given Point thereof.

15.

Let AB be the Axis, BC any Ordinate, take $DB = BF$, and draw the Ordinates De , Fo infinitely near C ; draw en and Cg parallel to AB , and produce C to k . Now by the Nature of Curvature, if the Curve be concave towards the Axis, in the Point C , and the Ordinates increase, then the Point o of the Curve will fall between k and g , and therefore the Increment go is less than nC , whence ok or $go - gk$ that is $go - nC$, will be negative; but ok is the second Moment of the Ordinate, and that is as the second Fluxion when Fk and De approach to, and coincide with BC ; therefore if the Curve be concave towards the Axis the second Fluxion of the Ordinate will be negative. And the contrary will happen if the Curve be convex towards the Axis. Wherefore

16.

Let the Axis $AB = x$, Ordinate $BC = y$, then compute the Value of $\ddot{y}$ by the Nature of the Curve, and substitute Numbers for all the Quantities if there be Occasion, then if its Value comes out negative it is concave in that Point, if affirmative it is convex towards the Axis.

X 2

Ex.

Ex. 16.

FIG. 17. *A Line being inflated into the Form of a Curve, and kept in that Form by a Force or Pressure acting perpendicularly upon every Point of the Curve; to find the Proportion between the Tension of the Curve, and the Force acting perpendicularly upon it.*

Let AB, BC be two equal Particles of the Curve and let the Force acting on the Particle B reduce it to the Position ABC ; compleat the Parallelogram $ABCE$, and draw AD, CD , perpendicular to AB, CB ; and then the four Points, A, B, C, D , will lie

in a Circle. Let AB or $BC = z$, AD or $CD = r$. Now by Mechanics the Point B is acted upon by three Forces BA, BC, BE ; BA or BC is the Tension of the Curve, and BE is the Force acting perpendicularly against the Curve, therefore these are to one another as BA to BE , or (by similar Triangles) as $\frac{1}{2} BD$ to

z . Let now the Points A, C , approach to B and coincide with it, and $\frac{1}{2} BD$ becomes $\frac{1}{2} r$, the Radius of

Curvature; and z is the Particle of the Curve whereon the Pressure acts: And therefore the Force acting perpendicularly on any Particle of the Curve is to the Tension of the Curve, as that Particle of the Curve is to the Radius of Curvature in that Point.

COR. Therefore if the Particle of the Curve and its Tension be given, the Force acting against the Particle is reciprocally as the Radius of Curvature that is, directly as the Curvature.

Ex. 17.

AF is a Curve Line supporting a Fluid; to find the F I G.
Nature of the Curve. 18.

Let the Axis of the Figure $CA=b$, $CD=x$, $DB=y$, $B=z$, let $\dot{z}$ be given: By reason of the Fluidity of the Water, the Tension of the Curve is equal in all points, and therefore by the forgoing Example, the Pressure at B is reciprocally as the Radius of Curvature $\ddot{x}$. But by the Laws of Hydrostatics the Pressure at B is as the Height DB or y , and (by Prob. V. Sect II.)

The Radius of Curvature is $\frac{\dot{z}\dot{y}}{\ddot{x}}$, therefore $\frac{1}{y} \propto \frac{\dot{z}\dot{y}}{\ddot{x}}$,

and assuming the given Quantity $\frac{aa}{2}$, then $\frac{aa}{2y} = \frac{\dot{z}\dot{y}}{\ddot{x}}$,

$\frac{aa\ddot{x}}{2} = y\dot{y}\dot{z}$, and the Fluent $aa\dot{x} = y^2\dot{z}$; but in

$\dot{x}=\dot{z}$ and $y=b$; therefore the Fluent corrected is

$\dot{x} - aa\dot{z} = yy - bb \times \dot{z}$, or $aa\dot{x} = aa + yy - bb \times$

$\dot{x} + \dot{y}^2$, which reduced gives $\dot{x} = \frac{aa - bb + yy \times \dot{y}}{\sqrt{2aabb + 2bb^2 - 2aa} y^2 - y^4}$

the Equation of the Curve sought.

Cor. Draw EK perpendicular, and IK parallel to CD , and also the Tangent EI ; then if $EK=p$, $EI=q$, $s = \text{Area } EAC$, which is as its Weight, then

(by Mechanics) $p : q :: s : \frac{qs}{p} = \text{Tension of the Curve}$,

hence (by the foregoing Example) since $y\dot{z} = \text{Pressure}$

at B , it will be $y\dot{z} : \frac{qs}{p} :: \dot{z} : \frac{\dot{z}\dot{y}}{\ddot{x}}$, and thence $\frac{qs\ddot{x}}{p}$

$y\dot{y}\dot{z} = \frac{aa}{2} \ddot{x}$, and therefore $\frac{2qs}{p} = aa$.

Ex.

Ex. 18.

FIG. To find an infinite Series, or several such, whose Sum may be had.

Assume $\dot{v} = \frac{\dot{x}}{1-x}$ or $\frac{\dot{x}}{1-x^2}$ or $\frac{\dot{x}}{1-x^3}$ &c. For

Example, let $\dot{v} = \frac{\dot{x}}{1-x} = 1 + x + x^2 + x^3$ &c.

$x : \dot{x}$, therefore $v = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$ &c.

Suppose the Fl: $v x^n \dot{x} = \frac{v x^{n+1}}{n+1} + q$; then by putting this into Fluxions, we shall find $q =$

$$\frac{-x^{n+1}}{n+1} \dot{v} = \frac{-x^{n+1}}{n+1} \times \frac{\dot{x}}{1-x} = \frac{x^{n+1} \dot{x}}{n+1 \times x - x^{n+2}}$$

$= \frac{1}{n+1} \times : x^n \dot{x} + x^{n-1} \dot{x} + x^{n-2} \dot{x} + x^{n-3} \dot{x}$, &c.

to $\frac{\dot{x}}{x-1}$ or $-\dot{v}$; therefore $q = \frac{1}{n+1} \times : \frac{x^{n+1}}{n+1}$

$\frac{x^n}{n} + \frac{x^{n-1}}{n-1}$ &c. $+ x - v$. Whence Fl: $v x^n \dot{x} =$

$\frac{v x^{n+1}}{n+1} + \frac{1}{n+1} \times : \frac{x^{n+1}}{n+1} + \frac{x^n}{n} + \frac{x^{n-1}}{n-1}$, &c. $+ x - v$.

But Fl: $v x^n \dot{x}$ is also $= F: x^n \dot{x} \times : x + \frac{x^2}{2} + \frac{x^3}{3}$
 $+ \frac{x^4}{4}$ &c. $= \frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{2 \times n+3} + \frac{x^{n+4}}{3 \times n+4}$, &c.
ad infinitum.

Therefore $\frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{2 \times n+3}$, &c. $= \frac{v x^{n+1}}{n+1}$
 $+ \frac{1}{n+1} \times : \frac{x^{n+1}}{n+1} + \frac{x^n}{n}$, &c. to x .

Put $\frac{vx^{n+1} - v}{n+1} = 0$, and then $x = 1$.

Therefore the foregoing Series will become

$$\frac{1}{+2} + \frac{1}{2 \times n+3} + \frac{1}{3 \times n+4} + \frac{1}{4 \times n+5} \text{ \&c.}$$

$$\text{ad infinitum} = \frac{1}{n+1} \times : \frac{1}{n+1} + \frac{1}{n} + \frac{1}{n-1} +$$

$$\frac{1}{-2} \text{ \&c. to } 1.$$

Or, which is the same Thing, $\frac{1}{n+1} \times : 1 + \frac{1}{2}$

$+\frac{1}{3} + \frac{1}{4} \text{ \&c. to } \frac{1}{n+1} = \text{Sum of the infinite Series}$

$$\frac{1}{+2} + \frac{1}{2 \times n+3} + \frac{1}{3 \times n+4} + \frac{1}{4 \times n+5} \text{ \&c.}$$

ad infinitum.

Now if n be put successively equal to any Integer Numbers, there will be had so many infinite Series, each of which will be equal to the Sum of a given finite Series.

For Example, suppose $n = 1$, then $\frac{1}{2} \times \overline{1 + \frac{1}{2}}$, or

$= \text{to the infinite Series } \frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} +$

$\frac{1}{4.6}, \text{ \&c.}$

Moreover, whatever be the Value of x , if v or

the F: $\frac{\dot{x}}{1-x}$ be known, we shall always have the fi-

nite Series $\frac{1}{n+1} \times : vx^{n+1} - v + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$

$\dots \text{ to } \frac{1}{n+1}x^{n+1} = \text{to the Series } \frac{x^{n+2}}{1 \times n+2} + \frac{x^{n+3}}{2 \times n+3}$

$+ \frac{x^{n+4}}{3 \times n+4} + \frac{x^{n+5}}{4 \times n+5}, \text{ \&c. ad infinitum.}$

The

FIG. The like may be done by assuming any other

Quantity for v ; as for Example, let $v = F: \frac{x}{1-x^3}$

$$\text{or } \dot{v} = \frac{\dot{x}}{1-x^3} = \dot{x} + x^3\dot{x} + x^6\dot{x} + x^9\dot{x}, \text{ \&c.}$$

$$v = x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + \frac{1}{10}x^{10}, \text{ \&c.}$$

Suppose, as before, $F: vx^n\dot{x} = \frac{vx^{n+1}}{n+1} + q$; then

$$\dot{q} = \frac{x^{n+1}}{n+1} \times \frac{\dot{x}}{x^3-1} = \frac{1}{n+1} \times x^{n-2}\dot{x} + x^{n-5}\dot{x}$$

$$+ \dot{x} - \dot{v}, \text{ and } q = \frac{1}{n+1} \times \left(\frac{x^{n-1}}{n-1} + \frac{x^{n-4}}{n-4} \right.$$

$$\left. \frac{x^{n-7}}{n-7} \dots + x - v = \frac{1}{n+1} \times \left(-v + x + \right.$$

$$\left. + \frac{x^7}{7} + \frac{x^{10}}{10} \dots + \frac{x^{n-1}}{n-1} \right).$$

Again, $F: vx^n\dot{x} = F: x^n\dot{x} \times x = x + \frac{1}{4}x^4 + \frac{1}{7}x^7$ \&c.

$$= \frac{x^{n+2}}{n+2} + \frac{x^{n+5}}{4n+5} + \frac{x^{n+8}}{7n+8}, \text{ \&c. therefore the}$$

$$\text{finite Series } \frac{1}{n+1} \times \left(vx^{n+1} - v + x + \frac{x^4}{4} \right.$$

$$\left. \frac{x^7}{7} \dots + \frac{x^{n-1}}{n-1} \right) = \text{the Sum of the Series } \frac{x^{n+2}}{n+2}$$

$$\frac{x^{n+5}}{4n+5} + \frac{x^{n+8}}{7n+8}, \text{ \&c. ad infinitum; where } n$$

any Number in this Series 2, 5, 8, 11, \&c. and in general,

$$\text{If } \dot{v} = \frac{\dot{x}}{1-x^m}, \text{ then } n \text{ must be such that } n+$$

may be divisible by m . And if you make $vx^{n+1} - v = 0$, and $x = 1$, as before, you may get as many particular Series as you please.

Ex. 19.

Express the Fluent of a simple Expression $ax^b\dot{x}$, by FIG. an infinite Series; and from thence to sum up that Series. 18.

Put $ax^b\dot{x} = \dot{F}$ and express the Index by two Parts

$$ax^{b+c}x^{-c}\dot{x}, \text{ then assume } F = \frac{ax^{b+c}}{b+c} x^{-c+1} + s,$$

then $\dot{F}$ or $ax^b\dot{x} = ax^b\dot{x} - c+1 \times \frac{a}{b+c} x^b\dot{x} + \dot{s}$, whence;

$$c-1 \times \frac{a}{b+c} x^b\dot{x} = c-1 \times Ax^b\dot{x}. \text{ Again,}$$

$$\text{Assume } s = \frac{c-1 \times A}{b+d} x^{b+d} x^{-d+1} + t = Bx^{b+d}$$

$x^{-d+1} + t$; whence $\dot{s} = d-1 \times Bx^b\dot{x}$.

$$\text{Assume } t = \frac{d-1 \times B}{b+e} x^{b+e} x^{-e+1} + v = Cx^{b+e}$$

$x^{-e+1} + v$; then $\dot{t} = e-1 \times Cx^b\dot{x}$.

$$\text{Assume } v = \frac{e-1}{b+f} Cx^{b+f} x^{-f+1} + w, \text{ \&c.}$$

$$\text{Thence } F \text{ or Fl: } ax^b\dot{x} = x^{b+1} \times : \frac{a}{b+c} + \frac{c-1}{b+d} A$$

$$+ \frac{d-1}{b+e} B + \frac{e-1}{b+f} C + \frac{f-1}{b+g} D + \text{\&c.} = \frac{ax^{b+1}}{b+1};$$

where $A, B, C, \text{ \&c.}$ are the preceding Terms.

Therefore taking $c, d, e, \text{ \&c.}$ any Quantities at Pleasure, the Series will be known, and the Sum of given.

1. Suppose $a=1, b, c, d, e, \text{ \&c.} = 2$. Then Fl:

$$x^2 = x^3 \times : \frac{1}{2} + \frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C, \text{ \&c.} = \frac{x^3}{3}. \text{ There-}$$

$$\text{fore } \frac{1}{2} + \frac{1}{2}A + \frac{1}{2}B, \text{ \&c.} = \frac{1}{2}.$$

Y

2. Sup-

F I G. 2. Suppose $b, c, d, e, \&c. = m$. Then Fl: ax^m

$$= x^{m+1} \times : \frac{a}{2m} + \frac{m-1}{2m} A + \frac{m-1}{2m} B + \frac{m-1}{2m} C$$

$$\&c. = \frac{ax^{m+1}}{m+1}; \text{ and } \frac{1}{m+1} = \frac{1}{2m} + \frac{m-1}{2m} A$$

$$\frac{m-1}{2m} B, \&c.$$

3. Let $a=1, b=n-1, c=2, d=3, e=4, f=5, \&c.$

$$\text{Then F: } x^{n-1} \dot{x} = x^n \times : \frac{1}{n+1} + \frac{1}{n+2} A + \frac{1}{n+3} B$$

$$+ \frac{1}{n+4} C, \&c. = \frac{x^n}{n}. \text{ Whence also } \frac{1}{n+1} A$$

$$+ \frac{1}{n+2} \times A + \frac{1}{n+3} B + \frac{1}{n+4} C, \&c. = \frac{1}{n}.$$

Ex. 20.

213. To find the Curve which a flexible Line QAR is put into by the Wind, or any Fluid moving against it.

Let AE be the Axis, and BE, Cf Ordinates infinitely near, Bd parallel to AE . Call AE, x ; EB, y ; AB, z ; $Bd, \dot{x}$; $Cd, \dot{y}$; $BC, \dot{z}$; and let $\dot{z}$ be given. The Force of the Fluid acting perpendicularly against the Particle of the Curve $\dot{z}$, is as the Quantity of the Fluid acting on it, and the Sine of Incidence

that is as $\dot{y} \times \frac{\dot{y}}{\dot{z}}$, or $\frac{\dot{y}^2}{\dot{z}}$, that is as $\frac{\dot{y}^2}{\dot{z}}$; And (by

Ex. 16. before) that Force will be as the Curvature in B , or reciprocally as the Radius of Curvature, that is (by Prob. V. Sect. II.) as $\frac{\dot{z}\dot{x}}{-\dot{y}}$; therefore

Therefore $\frac{\dot{y}^2}{z} \propto \frac{-\ddot{y}}{\dot{z}\dot{x}}$, or $\frac{\dot{y}^2}{z} = \frac{-a\ddot{y}}{\dot{x}}$; therefore $\frac{\dot{x}}{z} = \frac{F I G.}{213.}$

$\frac{-a\ddot{y}}{\dot{z}}$, and the Fluent is $\frac{x}{z} = \frac{a}{y}$; but in A , $\dot{y} = \dot{z}$, and $\dot{x} = 0$, therefore the corrected Fluent is $\frac{x}{z} = \frac{a}{y} - \frac{a}{z}$, whence $xy = az - ay$, therefore

$(a+x) \times \dot{y} = a\dot{z}$, and $(a+x)^2 \times \dot{z}^2 - \dot{x}^2 = a^2\dot{z}^2$, or $ax + xx \times \dot{z}^2 = (a+x)^2 \times \dot{x}^2$, and $\dot{z} = \frac{a+x \times \dot{x}}{\sqrt{2ax+xx}}$;

and the Fluent $z = \sqrt{2ax+xx}$, an Equation to the Catenary.

Ex. 21.

To find the Curve of Traction to a right Line. That is, if a heavy Body be placed at A , and a Cord AB of a given Length fastened to it, and the End B drawn along the right Line BD ; to find the Curve described by the Body A . 228.

Let AB be perpendicular to BD , and let AE be the Curve described by A . Draw the Ordinates EC , infinitely near each other, and $\perp$ to BD , and draw the Tangents ED , ed ; and $Do \perp$ to ed . Then put AB , ED or $ed = b$, $AE = z$, $BC = x$, $EC = y$, $CD = s$; then $bb = ss + yy$, and $2ss = -2yy$. And by similar Triangles, $y : b :: -\dot{y} : \dot{z}$, and $\dot{y} : \sqrt{bb - yy} :: -\dot{y} : \dot{x}$, whence

$\dot{z} = \frac{-b\dot{y}}{y} = \frac{b\dot{s}s}{bb - ss}$, and $\dot{x} = \frac{-\dot{y}}{y} \sqrt{bb - yy} = \frac{s^2\dot{s}}{bb - ss}$.

And by Form 2d and 4th, $z = -bL \times \text{Log: } y = \frac{bL}{2} \times \text{Log: } bb - ss$. But in A , $y = b$, or $s = 0$,

therefore the correct Fluent is $z = 2.30258b \times \text{Log: } \frac{bb}{bb - ss}$. Bu

FIG.
228.

But $\frac{-\dot{y}}{y} \sqrt{bb-yy} = \frac{-\dot{y} \times \overline{bb-yy}}{y\sqrt{bb-yy}} = bb$

$\frac{-y^{-2}\dot{y}}{\sqrt{-1+bb y^{-2}}} + \frac{y\dot{y}}{\sqrt{bb-yy}};$ and by Form
the 9th and 3d, the Fluent is $= bL$

Log: $\frac{b}{y} + \sqrt{-1 + \frac{bb}{yy}} : -\sqrt{b^2-y^2}.$

Also $\frac{s^2\dot{s}}{bb-ss} = -\dot{s} + \frac{bb\dot{s}}{bb-ss};$ and by Form

6th, the Fluent is $-s + \frac{bL}{2} \times \text{Log: } \frac{b+s}{b-s}.$ Therefore

$x = -\sqrt{bb-yy} + 2.30258b \times \text{Log: } \frac{b+\sqrt{bb-yy}}{y}$
 $= \frac{2.30258b}{2} \times \text{Log: } \frac{b+s}{b-s} : -s.$

Ex. 22.

229. To find the Curve of Traction to a Circle; or the Curve
EN described by a Body N, drawn along by a Cord
DN of a given Length, while the End D passes
through the Periphery of the Circle BAD.

Through N, n, two Points in the Curve infinitely
near, draw CA, Ca, and the two Tangents ND,
nd; draw DF $\perp$ to CA, and Do $\perp$ to nd;
and draw CD, and also CEB.

Then let CB=r, EB or ND=b, BA=x, CN=y,
EN=z, AD=v, DF=s, NF=P. Then in the
Triangle CDN, $rr = bb + yy + 2Py$, whence $P =$
 $\frac{rr - bb - yy}{2y} = \sqrt{bb - ss}.$ Then since ND=nd,

Nn or $\dot{z} = do$; also $\dot{v} = ad - AD = Dd - Aa =$
 $Dd - \dot{x}$, and $Dd = \dot{x} + \dot{v}$, and by the Nature of the

Circle $\dot{v} = \frac{rs}{\sqrt{rr - ss}}.$ By similar Sectors, $r : \dot{x} :: y :$

$y : nm = \frac{y\dot{x}}{r}$. And the Triangles DFN , nmN FIG.

229.

being similar, $s : b :: \frac{y\dot{x}}{r} : \dot{z} = \frac{by\dot{x}}{rs}$; and by

Trigonometry $b : s$ (S.DCN) $:: y : \frac{sy}{b} = S.CDN$

dDo ; and $r : \dot{x} + \dot{v}$ (Dd) $:: \frac{sy}{b} : (do \text{ or }) \dot{z} =$

$\frac{\dot{x} + sy\dot{v}}{rb} = \frac{by\dot{x}}{rs}$, and $\frac{s\dot{x} + s\dot{v}}{b} = \frac{b\dot{x}}{s}$, and $ss\dot{x}$

$+ ss\dot{v} = bb\dot{x}$; therefore $\dot{x} = \frac{ss\dot{v}}{bb - ss} =$

$\frac{rs^2\dot{s}}{bb - ss\sqrt{rr - ss}} = \frac{-r\dot{s}}{\sqrt{rr - ss}} + \frac{rbbs\dot{s}}{bb - ss\sqrt{rr - ss}}$.

By Form 10th, the Fluent of $\frac{rs}{\sqrt{rr - ss}} =$ Arch

whose Radius is r and Sine s , and is therefore =

Arch AD ; and $\frac{rbbs}{bb - ss\sqrt{rr - ss}} =$

$\frac{rbbs - 3s}{bb - ss\sqrt{rr - ss}} =$ (by Form the 23d)

$\frac{rb^2v^{-\frac{1}{2}}\dot{v}}{f \times p + \frac{bv}{f}} = \frac{rb^2}{-2rr} \times \frac{v^{-\frac{1}{2}}\dot{v}}{p + \frac{bb}{rr}v} =$

$\frac{bb}{2} \times \frac{v^{-\frac{1}{2}}\dot{v}}{-prr - bbv} = \frac{rb^2}{2} \times \frac{v^{-\frac{1}{2}}\dot{v}}{rr - bb - bbv}$;

whose Fluent (by Form the 6th) = $\frac{bbr}{2} \times$

$\frac{L}{\sqrt{rr - bb}} \times \text{Log: } \frac{\sqrt{rr - bb} + b\sqrt{v}}{\sqrt{rr - bb} - b\sqrt{v}} =$

brL

FIG. 229. $\frac{brL}{2\sqrt{rr-bb}} \times \text{Log: } \frac{\sqrt{rr-bb} + b\sqrt{\frac{rr-ss}{ss}}}{\sqrt{rr-bb} - b\sqrt{\frac{rr-ss}{ss}}}$

$$= \frac{brL}{2\sqrt{rr-bb}} \times \text{Log: } \frac{s\sqrt{rr-bb} + b\sqrt{rr-ss}}{s\sqrt{rr-bb} - b\sqrt{rr-ss}}$$

Therefore $x = \frac{brL}{2\sqrt{rr-bb}} \times \text{Log: } \frac{s\sqrt{rr-bb} + b\sqrt{rr-ss}}{s\sqrt{rr-bb} - b\sqrt{rr-ss}} - \text{Arch } AD.$

Again $P : b :: y : z = \frac{by}{P} = \frac{2by}{rr-bb-yy}$
 and $z = 2.30258b \times -\text{Log: } rr-bb-yy$: and corrected $z = 2.3025b \times \text{Log: } \frac{2rb-2bb}{rr-bb-yy}$.

SCHOLIUM.

Since $y = \frac{rr-bb-yy}{2by} z$, therefore when y

Maximum, y or $\frac{rr-bb-yy}{2by} z = 0$, and rr

$bb-yy = 0$; whence $y = \sqrt{rr-bb}$; then $P =$
 and $s = b$. Therefore the Curve can approach
 Circle no nearer than $r - \sqrt{rr-bb}$; therefore if
 Circle be described from the Center C, with
 Radius $\sqrt{rr-bb}$, this Circle will be an Asymp-
 to the Curve.

And let these few Examples here suffice, to the
 the Application of the Rules, to the Calculation
 particular Problems. We will now proceed to
 Resolution of more general Problems in Mathematic
 and Natural Philosophy.

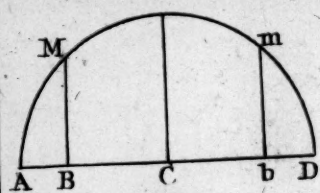
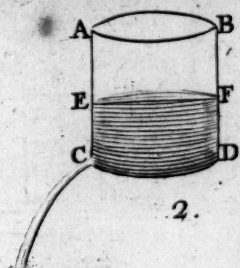
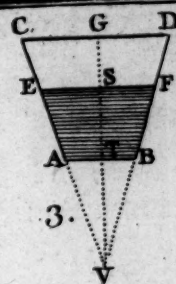


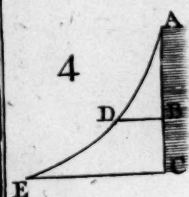
Fig. 1.



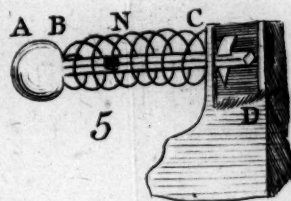
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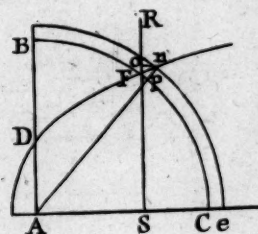
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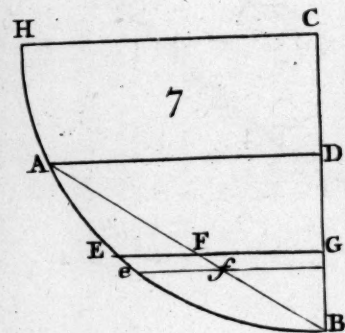
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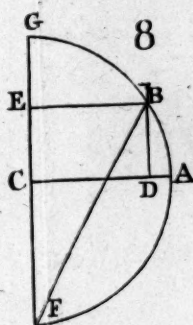
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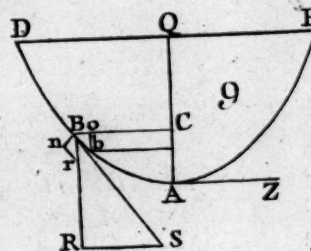
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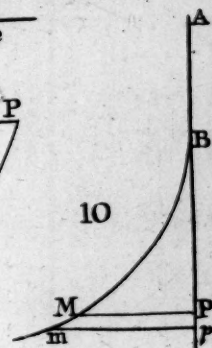
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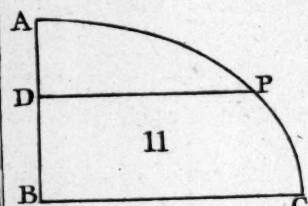
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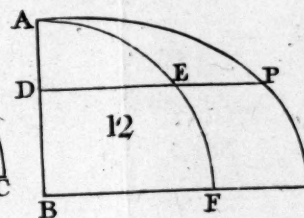
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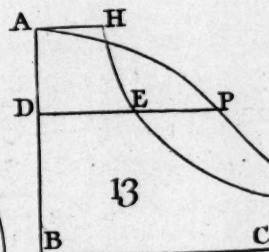
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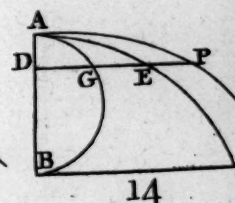
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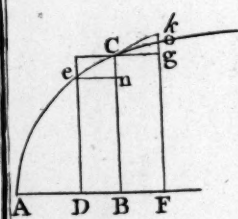
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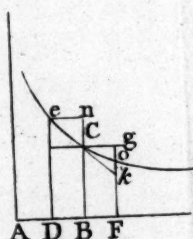
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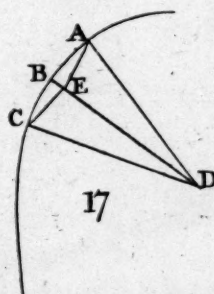
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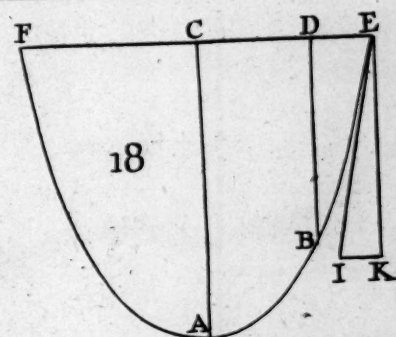
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17



18

S E C T. II.

*Investigation and Solution of some
of the most general and useful Pro-
blems in the Mathematics.*

P R O B. I.

*determine the Maxima and Minima of
Quantities.*

WHEN a Quantity is required to be the
greatest or least possible, under certain Con-
ditions, it is called a Maximum or Minimum; and
at that Moment it neither increases nor decreases, and
therefore its Fluxion is nothing. Now since any
Quantity is a Maximum or Minimum, when its
Fluxion is nothing, upon the Supposition of only one
variable Quantity therein, and the same is true when
any other Quantity alone is supposed variable: Con-
sequently when there are several variable Quantities
in the Maximum or Minimum, then each of these
Fluxions must be separately equal to nothing. There-

*Put the constant Quantity m for the Maximum or
Minimum required; and get an Equation involving m ,
by*

FIG. *by Help of which and other given Equations, exterminate as many of the variable Quantities as you please, there be several: Then put the remaining Equations in Fluxions, and for each Equation exterminate one Fluxion till you have but one Equation; in which (brought to one Side) make the Sum of all the Terms multiplied each particular Fluxion, separately $= 0$, and you will get as many Equations, which, together with those at first given, will determine all the unknown Quantities.*

Sometimes we must make the Logarithm of the Quantity a Maximum or Minimum, or the Log: of the Logarithm of it, &c. and put its Fluxion $= 0$. The Process is sometimes necessary in exponential Quantities. For if the Quantity itself be a Max. or Min. its Log will be so too.

2. But when the Fluxion of any single Quantity is found $= 0$, then that Quantity itself is either a Maximum or Minimum, or a standing Quantity. Or if an impossible Equation come out, the Quantity will have a Maximum or Minimum but what is infinite. Oftentimes the Equation will have several Roots, all which must be separately tried, to see which of them will answer the Conditions of the Question, and give the Maximum or Minimum required; and this is done by putting the single variable Quantity in the Maximum or Minimum equation to several successive Values expressed in Numbers.

230.

When the Nature of a Curve, as ECD , is required which shall contain some Maximum or Minimum, then suppose two Points of the Curve F and G to be given; then we must find an intermediate Point H so that the Part between F and G , or FH and GH may be a Maximum or Minimum. For if the Point H be not so situated, that the Part between FH and GH be a Maximum or Minimum, 'tis plain the whole cannot be so. Therefore,

3. Draw two Ordinates FH , GI infinitely near, intercepting an infinitely small given Part of the Max

or Minimum, and also of the given Quantity men-
 in the Problem, to find an intermediate Point C,
 the Ordinate PC, an arithmetic Mean between the
 two FH, GI. This ordinate divides these infi-
 nitely small Quantities into two Parts; then the Fluxion
 the Sum of each must be put = 0, which gives two
 Equations, from whence the Nature of the Curve will
 be had.

In Cases more compounded, where the Problem
 involves several Conditions, and the Nature of the
 Curve is required. Let EFD be the Curve, and
 any Point F be given therein, and let C be any
 other Point in the Curve, infinitely near F. Draw
 the Ordinates FH, CP; and Fm parallel to the
 Axis AB. To find the Situation of the Point F,
 that $A \times FC - B \times Fm$ may be = a Maximum or
 Minimum; supposing A, B, constant Quantities. Let
 $m = s$, $FC = t$, $Cm = n$, a given Quantity. Then

$\sqrt{nn+ss} - Bs = m$, in Fluxions $\frac{As\dot{s}}{\sqrt{nn+ss}} - B\dot{s} = 0$,

whence $As = B\sqrt{nn+ss} = Bt$, for the Nature of
 the Curve EFD. Now since HF (or y) is given in
 respect of the Point F; therefore A, B may be made
 of y and any given Quantities. And the same
 thing holds good of any other Point of the Curve.
 Therefore when $At - Bs = m$, the Equation of the
 Curve is $As = Bt$. But since $At - Bs = \text{Max. or}$
 Min. therefore the Sum of all the $At - Bs = \text{Max.}$
 Min. that is $F: A\dot{z} - F: B\dot{x} = m$, where FC or
 $\dot{z}$, Fm or $\dot{x} = \dot{z}$. And if either $F: A\dot{z}$ or $F:$
 $\dot{x}$ be given, the other is either a Maximum or Mi-
 nimum.

Whence the following Rule,

4. Let $AH = x$, $HF = y$, $EF = z$, and let $\dot{y}$ be
 given; and suppose $F: A\dot{z} - F: B\dot{x} = m$. Where A
 Z and

FIG. and *B* are given Quantities, or made up of *y* and given Quantities, or such as denote a Maximum or Minimum. All the Conditions of the Question being properly expressed collect all the Quantities affected with $\dot{x}$, for the Value *A*; and all the Quantities affected with $\dot{x}$, for the Value of *B*; then these Values being substituted in the Equation $A\dot{x} = B\dot{x}$, instead of *A* and *B*, gives the Equation of the Curve. And the same holds good if you make

$$F : \frac{\dot{z}}{B} - F : \frac{\dot{x}}{A} = \text{a Max. or Min.}$$

S C H O L.

Concerning curvilinear Spaces, it is in Effect the same Thing to seek the greatest Area contained under a given Perimeter, as to seek a given Area under the least Perimeter. The same will hold in Respect to Solids and their Surfaces.

EXAMPLE I.

To find $xz + yy$ a Minimum, so that $x + y + z = b$.

Put $xz + yy = m$, and expunging z , $bx - xx - xy + m = m$. In Fluxions $b\dot{x} - 2x\dot{x} - y\dot{x} - x\dot{y} + 2y\dot{y} = 0$, taking the homologous Terms, $b\dot{x} - 2x\dot{x} - y\dot{x} = 0$ and $2y\dot{y} - x\dot{y} = 0$, whence $2x + y = b$, and $x = 2y$; and therefore $y = \frac{1}{3}b$, $x = \frac{2}{3}b$, $z = \frac{1}{3}b$.

Or thus;

Since $xz + yy = m$, in Fluxions $x\dot{z} + z\dot{x} + 2y\dot{y} = 0$ also from the Equation $x + y + z = b$, we have $\dot{x} + \dot{y} + \dot{z} = 0$; and expunging $\dot{z}$, we get $-x\dot{x} - y\dot{x} + 2y\dot{y} = 0$, and therefore $z\dot{x} - x\dot{x} = 0$, or $z = x$; and $2y\dot{y} - x\dot{y} = 0$, or $x = 2y$, whence x, y, z are found the same as before.

Note, In this Example if x be given there will be a Minimum, but the Maximum is infinite; and if y be given there will be only a Maximum. Therefore in general there is no Maximum Minimum but what is infinite.

Ex. 2.

To find x the greatest in this Equation $x^3 - ax^2 + axy - y^3 = 0$. F I G.

For x write m , and the Equation is $m^3 - am^2 + amy - y^3 = 0$; In Fluxions $amy - 3y^2\dot{y} = 0$, whence $3y^2 = am$, which and the former Equation, m and y are determined.

Ex. 3.

To find a Cylinder of a given Solidity b , with the least whole Surface.

Let y = Altitude, x = Diameter of the Base, c = 1416, then $\frac{cx^2y}{4} = b$, and $\frac{cxx}{2}$ = Sum of the bases, cxy = convex Surface = $\frac{4b}{x}$; therefore $\frac{cx^2}{2} - \frac{4b}{x} = m$: In Fluxions $cx\dot{x} - \frac{4b\dot{x}}{xx} = 0$, which reduced gives $x = \sqrt[3]{\frac{4b}{c}}$, and thence $y = \sqrt[3]{\frac{4b}{c}} = x$.

Ex. 4.

In the Semicircle ABC , to find the Rectangle ADB a Maximum. 19.

Let $AC = a$, $AD = x$, $DB = \sqrt{ax - xx}$, then $\sqrt{ax - xx} = m$, or $ax^3 - x^4 = m^2$: In Fluxions $3ax^2\dot{x} - 4x^3\dot{x} = 0$, whence $x = \frac{3}{4}a$.

Ex. 5.

Given the Base AB and Perpendicular CD , in a Triangle ACB , to find the Angle ACB , the greatest possible. 20.

Bisect AB in E , and let $CD = p$, $AE = q$, $ED = y$; and by Trigonometry (CB) $\sqrt{pp + qq - 2qy + yy}$; (Radius)

F I G. (Radius) $R :: p : \frac{pR}{\sqrt{pp+qq-2qy+yy}} = S. <$
 20.

and (AC) $\sqrt{pp+q+y^2} : (S. < B) \frac{pR}{\sqrt{pp+q-y^2}}$

$:: 2q : \frac{2pqR}{\sqrt{pp+q+y^2} \times \sqrt{pp+q-y^2}} = S. <$

$= m$, and $p^4 + 2ppqq + q^4 + 2p^2y^2 - 2q^2y^2 + y^4 =$
 $\frac{4ppqqR^2}{mm}$: In Fluxions $4pp\dot{y}y - 4q\dot{y}y + 4y\dot{y} = 0$

and $y^3 = qq - pp \times y$, and one of the Roots is $y = 0$,
 the other Quantity $y^2 = qq - pp$ is an impossible
 Equation when p is bigger than q ; wherefore the
 Point D falls in E .

Ex. 6.

21. The Point P being given in the Transverse of the Ellipse
 ADR ; to find PB the nearest Distance to the Curve.

Let $AC = t$, $CD = c$, $AP = p$, $PR = q$, $PQ = x$

then $QB^2 = \frac{cc}{tt} \times \overline{pq+qx-px-xx}$, and $PB^2 =$

$\frac{cc}{tt} \times \overline{pq+qx-px-xx} + xx = m^2$: In Fluxions

$\frac{ccq\dot{x} - ccpx\dot{x} - 2ccx\dot{x}}{tt} + 2x\dot{x} = 0$; reduced $x =$

$\frac{cc \times \overline{p-q}}{2 \times tt - cc} = \frac{cc}{tt - cc} \times CP$.

Note, If CP be greater than $\frac{tt-cc}{t}$, then x will be
 greater than PR , which the Nature of the Question
 will not admit.

Ex. 7.

draw the Line EF to touch the Angle C of the $FIG.$ Rectangle $ABCD$, so that the Part, contain'd 22. between the Sides AB , AD produced, may be a Minimum.

Let $AD=a$, $AB=b$, $EB=x$, $EA=b+x$, then

$$F = \frac{ab+ax}{x}, \text{ and } EF = \sqrt{b+x^2} + \frac{aa}{xx} \times b+x^2.$$

$$m, \text{ or } bb+2bx+xx + \frac{aabb}{xx} + \frac{2a^2b}{x} + aa = m^2;$$

$$\text{this in Fluxions is } 2b\dot{x} + 2x\dot{x} - \frac{2aabb\dot{x}}{x^3} - \frac{2aabb\dot{x}}{xx}$$

$$\text{so; reduced is } x^4 + bx^3 - aabbx - aabb = 0, \text{ divided by } x+b=0, \text{ one of the Roots, and then } x^3 - aab = 0,$$

$$\text{and } x = \sqrt[3]{aab}, \text{ or from the other Equation } x = -b.$$

Or thus :

By similar Triangles $x : a :: b : \frac{ba}{x} = DF$, then

$$F = \sqrt{aa+xx} + \sqrt{bb + \frac{aabb}{xx}} = m : \text{ In Flux-}$$

$$\text{ions } \frac{x\dot{x}}{\sqrt{aa+xx}} - \frac{aabb\dot{x}}{x^3 \sqrt{bb + \frac{aabb}{xx}}} = 0; \text{ reduced } = aab.$$

Ex. 8.

find a Cone of the greatest Solidity under a given Convex Surface and Base b .

Let the Diameter of the Base $= x$, Side $= v$,

$$= 3.1416, \text{ then the Surface } = \frac{cxx}{4} + \frac{cxv}{2} = b, \text{ and}$$

Solidity

F I G. Solidity = $\frac{c^3 x}{12} \sqrt{v^2 - \frac{1}{4} x^2} = m$; and expunging

22.

$$\frac{c^3 x}{12} \sqrt{\frac{4bb}{c^2 x^2} - \frac{2b}{c}} = m, \text{ or } 4bbxx - 2bcx^4 = 144m^2$$

In Fluxions, $8bbx\dot{x} - 8bcx^3\dot{x} = 0$; reduced $x^2 = \frac{b}{c}$

hence $x = \sqrt{\frac{b}{c}}$, $v = \frac{3}{2} \sqrt{\frac{b}{c}}$, and Height = $\sqrt{\frac{b}{c}}$

and the Height, Base, and Side will be as $\sqrt{2}$, 1 and $1\frac{1}{2}$. And the same would be true if a Cone of given Solidity, under the least Surface was required.

Ex. 9.

To find y and x such, that $ay^3 - y^2x^2 + x^4$ may be Minimum.

This Equation in Fluxions is $3ay^2\dot{y} - 2x^2y\dot{y} - 2y^2x\dot{x} + 4x^3\dot{x} = 0$: And comparing the homologous Terms $3ay^2\dot{y} - 2x^2y\dot{y} = 0$, and $4x^3\dot{x} - 2y^2x\dot{x} = 0$; whence $3ay^2 = 2x^2y$, and $2x^2 = yy$, and therefore $3ay = yy$, and thence $y = 0$, or $y = 3a$; hence $x^2 = 0$, or $x^2 = \frac{3}{2}aa$.

Note, if y be given, the Quantity $ay^3 - y^2x^2 + x^4$ has a Minimum, but the Maximum is infinite; and if x be given, it has a Minimum, but the Maximum is also infinite: Therefore if neither be given, it has a Minimum, but no Maximum.

Ex. 10.

To find $xy^2u^3z^4$ a Maximum so that $x + y + u + z = b$

Here $xy^2u^3z^4 = m$, and expunging x , $y^2u^3z^4 = \frac{m}{x}$

$$b - y - u - z = m, \text{ or } b - y - u - z = \frac{m}{y^2u^3z^4}$$

$$\text{In Fluxions, } -\dot{y} - \dot{u} - \dot{z} = -\frac{2m\dot{y}}{y^3u^3z^4} - \frac{3m\dot{u}}{y^2u^4z^4} - \frac{4m\dot{z}}{y^2u^3z^5}$$

ing $\frac{m\dot{z}}{u^3z^5}$; and collecting separately the homologous F I G.

quantities, $\dot{y} = \frac{2m\dot{y}}{y^3u^3z^4}$, $\dot{u} = \frac{3m\dot{u}}{y^2u^4z^4}$, $\dot{z} = \frac{4m\dot{z}}{y^2u^3z^5}$;

hence $\frac{m}{y^2u^3z^4} = \frac{y}{2} = \frac{u}{3} = \frac{z}{4} = b - y - u - z$;

therefore $u = \frac{2}{3}y$, $z = \frac{4}{3}u = 2y$, and therefore $\frac{1}{2}y = b - \frac{2}{3}y$, or $y = \frac{2}{5}b$; hence $x = \frac{1}{10}b$, $y = \frac{2}{5}b$, $u = \frac{4}{5}b$, $z = \frac{4}{5}b$.

EX. II.

To find a Trapezoid ABCD of a given Area b , whose 23.
two Sides and Base $AB + BC + CD$ shall be the least possible.

Let $BC = x$, $BA = y$, $CD = u$, perpendicular BE
 $CF = z$. Then $y + x + u = m$, and $z\sqrt{yy - zz} +$
 $\sqrt{uu - zz} + 2zx = 2b$, or (dividing by z) $\sqrt{yy - zz}$
 $+\sqrt{uu - zz} + 2x = \frac{2b}{z}$. Put these Equations into

fluxions, and $\dot{y} + \dot{x} + \dot{u} = 0$, and $\frac{y\dot{y} - z\dot{z}}{\sqrt{yy - zz}} + \frac{u\dot{u} - z\dot{z}}{\sqrt{uu - zz}}$

$+ 2\dot{x} = \frac{-2b\dot{z}}{zz}$; and expunging $\dot{x}$, $\frac{y\dot{y} - z\dot{z}}{\sqrt{yy - zz}} +$

$\frac{u\dot{u} - z\dot{z}}{\sqrt{uu - zz}} - 2\dot{y} - 2\dot{u} = \frac{-2b\dot{z}}{zz}$; whence $\frac{y\dot{y}}{\sqrt{yy - zz}}$

$- 2\dot{y} = 0$, and $4z^2 = 3y^2$. Also $\frac{u\dot{u}}{\sqrt{uu - zz}} - 2\dot{u}$

$= 0$, and thence $4z^2 = 3uu$. Therefore $3y^2 = 3u^2$,

and $y = u$. Lastly $\frac{-z\dot{z}}{\sqrt{yy - zz}} - \frac{z\dot{z}}{\sqrt{uu - zz}} = \frac{-2b\dot{z}}{zz}$,

that

FIG. that is $\frac{z^3}{\sqrt{yy-zz}} + \frac{z^3}{\sqrt{uu-zz}} = 2b$; and expunging

23.

$$z \text{ and } u, \frac{z^3}{\frac{1}{2}y} + \frac{z^3}{\frac{1}{2}u} = \frac{4z^3}{y} = \frac{1}{2}y^2\sqrt{3} = 2b, \text{ and}$$

$$= 2\sqrt{\frac{bb}{27}} = \frac{2}{3}\sqrt[4]{3bb} = u, \text{ and } z = \frac{b}{z} - \frac{1}{2}\sqrt{yy-zz}$$

$$- \frac{1}{2}\sqrt{uu-zz} = \frac{2}{3}\sqrt[4]{3bb}; \text{ and } z = \sqrt[4]{\frac{bb}{3}}. \text{ Hence}$$

$AB = 2AE = 2FD$, and $AB = BC = CD$: And the Figure is inscribed in a Semicircle, whose Diameter is AD .

Ex. 12.

24. *Given the Velocity of a Projectile, and its Height A above the Horizon: To find the Angle of Elevation LAG , to throw it to the greatest Distance possible on the horizontal Plane BC .*

Let $s = AD$ the Space described in a given Time $d = DE$ the Space described by a falling Body in the same Time; $b = AB$; and let AG be the Line of Projection, AL parallel to BC , $GL = x$. Then by

the Laws of falling Bodies, $d : b + x :: ss : \frac{bss + sxx}{d}$

$$= AG^2. \text{ Therefore } AL = \sqrt{\frac{bss + sxx}{d} - xx} = B$$

$$= m, \text{ or } bss + sxx - dxx = dmm; \text{ in Fluxions, } ssx -$$

$$2dxx = 0, \text{ whence } x = \frac{ss}{2d}. \text{ Therefore } AG =$$

$$\frac{s}{2d}\sqrt{4bd + 2ss}, AL = \frac{s}{2d}\sqrt{4bd + ss}, AC = b +$$

$$= CG; \text{ whence the Angle } FAC \text{ is bisected by the}$$

Line of Elevation or Direction AG ; and $\sqrt{4uu +$

$: s :: (AL : GL ::) \text{ Rad. : Tangent of the Angle of Elevation } LAG.$

Ex

Ex. 13.

If the Weight w be given, and the Radii of the Wheel FIG.
 and Axel BA , AC ; to find the weight y , to be raised 231.
 by the Descent of w , so that y may receive the greatest
 Motion possible in a given Time.

Let $AC=a$, $AB=b$, and put $p+q=w$, so that the
 Part p may just balance y upon the Wheel, and there-
 fore $bp=ay$. Let $2b$ =Velocity gain'd by Gravity in
 the Second, and v =Velocity gain'd by y in its Ascent,

in the same Time; and then $\frac{b}{a}v$ = Velocity of w ;

and the Motion which would be generated in q by its
 Gravity will be $=2bq$. But since the Motion of the
 Bodies w , y , is generated by the same Gravity of q ,
 therefore that Motion will be equal to the former;
 or when the Force is given, the Motion generated,

in a given Time, will be given. But $vy + \frac{b}{a}vw$

= Motion generated in w and y , therefore $vy +$
 $vw = 2bq$. But $q = w - p = w - \frac{ay}{b}$;

$$\text{hence } v = \frac{2bq}{\frac{b}{a}w + y} = \frac{2b \times w - \frac{a}{b}y}{\frac{b}{a}w + y} =$$

$$\frac{ab \times bw - ay}{b \times bw + ay}; \text{ and } vy = \frac{2aby \times bw - ay}{b \times bw + ay} = m,$$

by the Question, or $\frac{bwy - ayy}{bw + ay} = \frac{bm}{2ab}$. In

fluxions, $bwy - 2ayy \times bw + ay - ay \times vwy - ayy$
 $= 0$, and $bbww - 2abwy - aayy = 0$; which re-

A a

duced

F I G. duced is $y = \frac{\sqrt{2-1}}{1} \times \frac{bw}{a} = \frac{.414bw}{a}$. And

hence is found $v = \frac{2ba}{b} \times \sqrt{2-1} = .414bw$. *and the velocity of w = .414bw*

And if $b = a$, then $y = .414w$.

SCHOLIUM.

If the Brachium a was sought, all the rest being given, we should come to the same Conclusion, that is, $a = \frac{.414bw}{y}$.

Ex. 14.

232. *The Arch AD being given, to divide it into three Parts AB, BC, CD; so that the Product of any Power of the Sines $BE^n \times CI^n \times DO^n$, may be the greatest possible.*

Let Radius $RA=1$, Arches $AB=A$, $BC=B$, $CD=C$, and $AD=g$; Sines $BE=x$, $CI=y$, $DO=z$; Cosines $RE=X$, $RI=Y$, $RO=S$; $Bq=A$, $Bp=B$, and let $x^m y^n z^r = \text{Max.}$ by similar Triangles $X(RE)$

$1 (RB) :: x (Bp) : A (Bq) = \frac{x}{X}$; likewise $B = \frac{y}{Y}$, and $C = \frac{z}{S}$.

And since $A+B+C=g$, there

fore $A+B+C=g$, or $\frac{x}{X} + \frac{y}{Y} + \frac{z}{S} = g$.

And since $x^m y^n z^r = \text{Max.}$ therefore $mx^{m-1}xy^nz^r = 0$

$ny^{n-1}yx^mz^r + rz^{r-1}x^my^n = 0$, or $\frac{mx}{x} + \frac{ny}{y} + \frac{rz}{z} = 0$

From these two Equations we get $-z = S \times$

$\frac{x}{X} + \frac{y}{Y}$; and $-z = \frac{s}{r} \times \left(\frac{mx}{x} + \frac{ny}{y} \right)$: there

fore $\frac{Sx}{X} + \frac{Sy}{Y} = \frac{smx}{rx} + \frac{sn y}{ry}$. Hence (a

cordin

According to Art. 1 of the Rule) we shall have $\frac{S\dot{x}}{X} = F I G.$
^{232.}

$\frac{sm\dot{x}}{rx}$, and $\frac{Sy}{r} = \frac{sn\dot{y}}{ry}$; therefore $\frac{rx}{X} = \frac{ms}{s}$,

and $\frac{ry}{r} = \frac{ns}{s}$. Hence $\frac{x}{X} : \frac{s}{s} :: m:r$, and

$\frac{y}{r} : \frac{s}{s} :: r:n$; and *ex e quo*, $\frac{x}{X} : \frac{y}{r} :: m:n$.

But by Trigonometry, $\frac{x}{X}$, $\frac{y}{r}$ and $\frac{s}{s}$ are the

Tangents of the Arches A, B, C ; therefore the Tangents of the Arches A, B, C , are respectively the Indices of the Powers m, n, r . And the same holds true for any Number of Arches whatever.

COR. 1. If there be only two Arches A, B ; then $+n : m-n :: \text{Tan. } A + T.B : T.A - T.B ::$
 (by Trigonometry) $S. A + B : S. A - B$; that is
 $+n : m-n :: S. \text{Sum of the Arches } A, B :$
 of the Difference ; whence the Arches will be known.

Or thus. Let Tangent of $A = mv$, of $B = nv$, of $A+B = t$. Then by Trigonometry $\frac{m+n \times v}{1 - mnvv} = t$,
 quadratic Equation for v .

COR. 2. If there be three Arches A, B, C ; let their Tangents be mv, nv, rv , and $\text{Tan. Sum} = t$;

$\text{Tan. } A+B = z$. Then by Trigonometry $\frac{m+n \times v}{1 - mnvv}$

$= z$, and $\frac{z + rv}{1 - zrv} = t$; reduced $\frac{m+n+r.v - mnrv^3}{1 - mn + mr + nr.vv}$

$= t$, a cubic Equation for v .

And if there be 4 Arches, you'll have a 4th Power;
 5 Arches, a 5th Power of v , &c.

A a 2

Ex.

Ex. 15.

FIG. To find x^s a Maximum, putting $s = \frac{1}{x}$.

Let $X = \text{Hyp. Log. of } x$. Then since x^s is a Maximum, its Log. (by Art. 1.) is a Maximum; that is, sX or $\frac{X}{x}$ is a Maximum: and the Log.

$\frac{X}{x}$ is also a Maximum; that is, $\text{Log. } X - \text{Log. } x$

or $\text{Log. } X - xX$ is a Max. and its Fluxion, $\frac{\dot{X}}{x} - X - x\dot{X}$

$= 0$. But $\dot{X} = \frac{\dot{x}}{x}$, (see the next Prob.)

therefore $\frac{\dot{x}}{xX} - \dot{x} - X\dot{x} = 0$; whence $X^2x + Xx = 1$.

Let $x = 1 + v$; then $X = v - \frac{v^2}{2} + \frac{v^3}{3} - \frac{v^4}{4} + \text{Ec.}$ (by Example 3, of the next Prob.); therefore

$\frac{v - \frac{v^2}{2} + \frac{v^3}{3} - \frac{v^4}{4} + \text{Ec.}}{1 + v} + v - \frac{v^2}{2} + \frac{v^3}{3} - \frac{v^4}{4} + \text{Ec.} \times$
 $\frac{1}{1 + v} = 1$. That is, $v + \frac{1}{2}v^2 - \frac{1}{6}v^3 + \text{Ec.} = 1$.
 Whence, by Reversion, $v = .56$, and $x = 1.56$.

But because this does not converge fast, put $x = 1.56 + v$; $l = \text{Hyp. Log. } n (= 2.302585)$
 $\times \text{vulgar Log. } n$, then will $X = l + \frac{v}{n} - \frac{vv}{2n^2}$

+ Ec. whence we shall have $l + \frac{v}{n} - \frac{vv}{2n^2}$

$$1 + \frac{v}{n} - \frac{vv}{2nn} : \times n + v : = 1. \text{ That is, F I G.}$$

$$1 \times nl + \frac{l+1}{1+l}^2 + l \times v + \frac{l+\frac{3}{2}}{n} vv, \&c. = 1.$$

In Numbers.

$$.0021921 + 2.5318022v + 1.246593vv, \&c. = 1.$$

$$\text{Or } 2.031v + vv = -.0017586.$$

$$\text{Hence } v = -.0008661; \text{ and } n + v \text{ or } x = 1.5591339.$$

Otherwise thus.

Let $l + p = X$; n = Number of the Hyp.
 logarithm l , as before. Then x (or the Number of
 the Hyp. Logarithm $l + p$ or of X is) $= n \times : 1 +$
 $+\frac{1}{2}pp + \frac{1}{6}p^3 \&c.$ (by Schol. 2. of the next Prob.)
 Hence the Equation $\overline{XX} + X \times x = 1$, becomes

$$1 + p + l + p \times \text{into } n + np + \frac{1}{2}npp \&c. = 1.$$

That is,

$$\left. \begin{aligned} 1 \times l + \frac{l+1}{1+l}^2 \end{aligned} \right\} p \left. \begin{aligned} + \frac{1}{2}ll \\ + 2\frac{1}{2}l \\ + 2 \end{aligned} \right\} p^2 : \times n = 1.$$

In Numbers.

$$.002192 + 3.949611p + 5.008515pp \&c. = 1.$$

$$\text{Or } .78858p + pp = -.00043768.$$

$$\text{Whence } p = -.0005549, \text{ and } l + p = .4441309$$

$$\text{Hyp. Log. } x; \text{ or } \frac{l+p}{1+p} \times .4342945 = \text{vulgar Log.}$$

$$x \text{ (See Cor. to Ex. 8th of the next Prob.)} =$$

$$.1928836; \text{ whence } x = 1.559134.$$

Note x^s or $x^{\frac{1}{x}}$ has no Minimum.

Here

F I G. Here follow some Examples of finding the Nature of Curves, that require some *Maximum* or *Minimum*.

Ex. 16.

230. To determine the Nature of the Curve EFD, so that the Length of the Arch AD being given, the Area ABF shall be a Maximum.

Let HFGI be an infinitely small given Part of the Area, and C an intermediate Point (according to the Rule Art. 3.) and draw the Ordinates FH, GI, and CP; and Fm, Cn parallel to AB; and suppose Cm = Gn.

Put the given Quantities FH = p, CP = q, Cm = Gn = n; and the variable ones Fm = v, Cn = s.

Then by the Nature of the Problem, FC + CG = given Quantity = $\sqrt{vv + nn} + \sqrt{ss + nn}$: Therefore

$$\frac{v\dot{v}}{\sqrt{vv + nn}} + \frac{s\dot{s}}{\sqrt{ss + nn}} = 0. \text{ And because}$$

the Area pv + qs is a Max. = m, therefore p\dot{v} + q\dot{s} = 0, or p\dot{v} = -q\dot{s}, whence

$$\frac{-s\dot{s}}{\sqrt{ss + nn}} = \frac{sp\dot{v}}{q\sqrt{ss + nn}}; \text{ that is (making}$$

the Ordinates alike affected) $\frac{v}{p\sqrt{vv + nn}} =$

$$\frac{s}{q\sqrt{ss + nn}}, \text{ or } \frac{Fm}{FH \times FC} = \frac{Cn}{CP \times CG}; \text{ consequently (any one as) } \frac{Fm}{FH \times FC} \text{ is an invariable}$$

Quantity. Therefore if AH = x, HF = y, EF = z,

then $\frac{x\dot{x}}{y\dot{z}} = \frac{1}{a}$, or ax = yz, for the Nature of the Curve, which will be a Circle.

§. II. of FLUXIONS.

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If the Length of the Arch be given, and the Solid to F I G. a Maximum; then will $p^2v + q^2s = m$; whence 230.

$\dot{v} = -q^2\dot{s}$; which gives $\frac{v^3}{pp\sqrt{vv+nn}}$ or $\frac{\dot{x}}{yy\dot{z}}$ a given Quantity, or $aa\dot{x} = yy\dot{z}$.

If the Surface AD be given, to have the Solid a Maximum. Then you'll have $p\sqrt{vv+nn} + q\sqrt{ss+nn}$ a given Quantity; and $ppv + qqs = m$; which gives $\frac{v}{p\sqrt{vv+nn}}$ or $\frac{\dot{x}}{y\dot{z}}$ a given Quantity, that $aa\dot{x} = y\dot{z}$, as in the first. The same Equation will be found, if the Solid be given, and the Surface a Minimum.

Ex. 17.

Find the Nature of the Curve ERD, which generates the Solid of least Resistance; whose Length and greatest Diameter are given. 233.

Let CP, an infinitely small Part of the Axis, be given; draw the Ordinates CR, PT; and the intermediate one QF; and let $PT=p$, $QF=q$, $Tm=v$, $Fm=s$, $Fm=Rn=n$.

Then by Mechanics, the Force of the Fluid against the Part TF of the Surface, will be as the Cube of the Sine of Incidence, and magnitude of the

Surface, that is as $\frac{n^3}{TF^3} \times p \times FT$, or $\frac{pn^3}{FT^2}$. Therefore

the Force of the Fluid upon the two Parts TF,

R, will be $\frac{n^3p}{n^2 + v^2} + \frac{n^3q}{nn + ss} = m$, a Minimum.

Therefore $\frac{-2n^3pv\dot{v}}{(n^2 + v^2)^2} - \frac{2n^3q\dot{s}}{(nn + ss)^2} = 0$. And

since $s + v =$ a given Quantity, therefore $\dot{v} = -\dot{s}$, whence

FIG. whence $\frac{n^3pv}{nn+vv} = \frac{n^3qs}{nn+ss}$, or $\frac{n^3pv}{TF^4} = a$ given
233.

Quantity; that is, putting $p=y$, $v=\dot{x}$, $\frac{y^3\dot{x}}{z^4} =$
or $y\dot{y}^3\dot{x} = az^4$.

When the Area of the generating Plane ABED, and the greatest Diameter BE are given; then $pv+qs$ is a given Quantity, and $p\dot{v} = -q\dot{s}$; which gives

$$\frac{n^3v}{nn+vv} = \frac{n^3s}{nn+ss}, \text{ or } \frac{y^3\dot{x}}{z^4} = a.$$

If its Bulk AE, and greatest Diameter be given, then $p^2v+qqs = a$ given Quantity, or $p^2\dot{v} = -q^2\dot{s}$

which gives $\frac{n^3v}{p \times TF^4} = \frac{n^3s}{q \times FR^4}$, or $\frac{y^3\dot{x}}{yz^4} = a$.

Ex. 18.

234. To find the Nature of the Curve ADK, wherein a Body will descend from A to K in the shortest Time; Points A, K being given.

Suppose BE an infinitely small given Part of the Axis, and draw the Ordinates BD, EH, and the middle one CM; and let the invariable Lines $BD=p$, $CM=q$, $OM=n=gH$; and the variable ones $BC=v$, $CE=s$; then the Velocity at D will be $\sqrt{BD}$, and at M as $\sqrt{CM}$; and the Times being the Spaces directly and Velocities reciprocally, therefore

$$\frac{DM}{\sqrt{BD}} \text{ will be as the Time of describing } DM$$

$$\text{and } \frac{MH}{\sqrt{CM}} \text{ as the Time in } MH; \text{ therefore } \frac{\sqrt{vv+nn}}{\sqrt{q}}$$

$$+ \frac{\sqrt{ss+nn}}{\sqrt{q}} = m, \text{ also } v+s = a \text{ given Quantity}$$

when

hence $\frac{v\dot{v}}{\sqrt{p}\sqrt{vv+nn}} + \frac{s\dot{s}}{\sqrt{q}\sqrt{ss+nn}} = 0$, and FIG. 234.

$+s=0$; therefore we have $\frac{v}{\sqrt{p}\sqrt{vv+nn}} =$

$\frac{s}{\sqrt{q}\sqrt{ss+nn}}$; that is (putting $AB=x$, $BD=y$)

$\frac{\dot{x}}{\sqrt{y}} = \frac{1}{\sqrt{a}}$, a given Quantity; whence $\dot{x}\sqrt{a} = \dot{y}$, or $a\dot{x}^2 = y\dot{z}^2 = y\dot{x}^2 + y\dot{y}^2$, and $\dot{x}\sqrt{a-y} = \dot{y}\sqrt{y}$, which denotes a Cycloid.

If the Velocity at D be as any Power or Sums of Powers of BD , which call R ; it will appear the same Way, that $\frac{\dot{x}}{R\dot{z}} =$ a given Quantity.

If the Area ARK be given; then $pv+qs$ is given,

or $p\dot{v} = -q\dot{s}$, whence $\frac{v\dot{v}}{p^{\frac{3}{2}} \times DM} = \frac{s\dot{s}}{q^{\frac{3}{2}} \times MH}$, or

$\frac{\dot{x}}{y^{\frac{3}{2}} \dot{z}} = \frac{1}{a^{\frac{3}{2}}}$; and if the Velocity at D be as

BD^m , then $\frac{\dot{x}}{y^{m+1} \dot{z}} = \frac{1}{a^{m+1}}$.

Ex. 19.

Given CA , CT and the Length of the Curve ADT ; 235
to find the Nature of it, so that the Area ACT may be a Maximum.

Let the infinitely small Parts of the Area BCE , and of the Curve BE be given; and draw the mean Ordinate CD , and put $CD=p$, $CE=q$; $Bo=v$, $Dg=s$, Do or $Eg=n$. Then $\sqrt{vv+nn} + \sqrt{ss+nn}$ is a given Quantity, and $\frac{pv+qs}{2} = m$, a Maximum.

B b

There-

FIG. Therefore $\frac{v\dot{v}}{\sqrt{vv+nn}} + \frac{s\dot{s}}{\sqrt{ss+nn}} = 0$, and $p\dot{v} =$
235.

$$-q\dot{s}; \text{ whence } \frac{v\dot{v}}{\sqrt{vv+nn}} = \frac{sp\dot{v}}{q\sqrt{ss+nn}};$$

$$\frac{v}{p \times BD} = \frac{s}{q \times DE} : \text{ Put } CB = y, Bo = \dot{x}, Do = \dot{y}$$

then $\frac{\dot{x}}{y\dot{z}} = \frac{1}{a}$, a given Quantity, or $a\dot{x} = y\dot{z}$, a
Equation to the Circle.

Ex. 20.

To find the Nature of the Curve *ADT*, in which a Body
will move from *T* to *A* in the least Time possible; the
Velocity at any Point *E* being as CE^m , the m^{th} Power
of the Distance.

Take *ECB* an infinitely small given Angle, and
draw the mean Ordinate *CD*; and put $CB = p$, $CD = q$,
 $Bo = v$, $Dg = s$, Do or $Eg = n$. Then the Time be-
ing as the Space directly and Velocity reciprocally

$$\text{we shall have } \frac{\sqrt{vv+nn}}{p^m} + \frac{\sqrt{ss+nn}}{q^m} =$$

$$\text{Whence } \frac{v\dot{v}}{p^m \sqrt{vv+nn}} + \frac{s\dot{s}}{q^m \sqrt{ss+nn}} = 0, \text{ and}$$

since the Angle *ECB* is given, if $\text{Rad.} = 1$, then

$$\frac{v}{p} = \angle BCD, \frac{s}{q} = \angle DCE, \text{ and } \frac{v}{p} + \frac{s}{q} =$$

given Quantity, and $\frac{\dot{v}}{p} = \frac{-\dot{s}}{q}$; therefore

$$\frac{v\dot{v}}{p^m \sqrt{v^2+n^2}} = \frac{-s\dot{s}}{q^m \sqrt{ss+nn}} = \frac{sq\dot{v}}{pq^m \sqrt{ss+nn}}$$

or $\frac{v}{p^{m-1} \times BD} = \frac{s}{q^{m-1} \times DE}$; that is $\frac{\dot{x}}{y^{m-1} \dot{z}}$
 $= \frac{1}{a^{m-1}}$, a given Quantity.

If $m=0$, AD is a right Line, where a is the Perpendicular from C .

If $m=1$, then $\frac{\dot{x}}{\dot{z}}$ is a constant Quantity, and the Curve is a Log. Spiral.

If $m=2$, then $a\dot{x}=y\dot{z}$, the Equation of a Circle.

If $R = \text{any Sums of the Powers of } y$, representing the Velocity; then will $\frac{y\dot{x}}{R\dot{z}} = \text{a constant Quantity}$ as before.

Here follow some Examples of Curves, where more Conditions are concerned in the Question.

Ex. 21.

To determine the Nature of the Curve EFD , so that the Points E, D being given, as also the Length of the Arch ED ; the Area ABD shall be a Maximum. 230.

Let $AH=x$, $HF=y$, $EF=z$, then since the Points E, D , are given, the Length of the Axis AB is given, therefore the Axis $AB=F: a\dot{x}$; also the Curve $ED=F: b\dot{z}$; and the Area of the Figure $= F: cy\dot{x}$. Therefore collecting the Quantities that are affected with $\dot{z}$, and also those that are affected with $\dot{x}$; we shall have $F: A\dot{z} - F: B\dot{x} = F: b\dot{z} - F: \overline{a+cy} \times \dot{x}$; whence $A=b$, $B=a+cy$, therefore by the Rule Art. 4. $A\dot{x}=B\dot{z}$ becomes $b\dot{x}=a\dot{z}+cy\dot{z}$, for the Nature of the Curve, which denotes the Arch of a Circle.

If the Solid generated about AB, is required to be
 F. I. G. *Maximum*; then the Solid = $F: dyx$; and $A =$
 230. $B = a + dy$, and $bx = az + dyx$ is the Nature of the
 Curve.

If the Surface generated by ED is given, to have the
Solid a Maximum; then the Surface = $F: gy$
 whence $A = b + gy$, $B = dy$; and $gyx = az + dyx$
 for the Nature of the Curve. Here the Coefficients
 a, b, c, d, g are constant Quantities, affirmative or
 negative, to be determined by the Circumstances of
 the Question.

Ex. 22.

233. *To find the Nature of the Curve ERD, which generates*
the Solid of least Resistance moving in a Fluid; whose
Length, greatest Diameter, and solid Content is given.

Let $AP = x$, $PT = y$, $DT = z$; then Axis $AB =$
 $F: ax$, and Solidity = $F: byx$, then by the Reasoning
 in Ex. 16. the Force of the Fluid upon the An-
 nulus $PTFQ$ is $\frac{ryy^3}{z^3} \times z$, which is a Minimum

and the whole Resistance = $F: \frac{ryy^3}{z^3} \times z$. And col-

lecting the Quantities affected with z , and also those

affected with x , and we have $F: Az - F: Bx =$
 $F: \frac{ryy^3}{z^3} \times z - F: ax + byx$; whence $A = \frac{ryy^3}{z^3}$

$B = a + by$. Therefore $(Ax = Bz) \frac{ryy^3}{z^3} x = az +$
 $byyz$, for the Nature of the Curve.

If instead of the Solidity, the Surface is given; then
 the Surface = $F: cyz$. Therefore $A = \frac{ryy^3}{z^3} + c$

and $B = a$. Then $(Ax = Bz) \frac{ryy^3}{z^3} x + cyx = az$
 for the Nature of the Curve DTE.

If both the Surface and Solidity be given; then FIG.

$$= \frac{ryy^3}{z^3} + cy, B=a+byy. \text{ And } (A\dot{x}=B\dot{z}) \frac{ryy^3}{z^3} \dot{x} \\ cy\dot{x}=a\dot{z}+byy\dot{z}, \text{ for the Nature of the Curve.}$$

Ex. 23.

find the Nature of the Curve ADK, wherein a Body will descend from A to K, in the shortest Time; the Points A, K being given, and the Length of the Curve ADK.

234.

Let $AB=x$, $BD=y$, $AD=z$. The Velocity at is as $\sqrt{y}$, and the Time of describing DM or $\dot{z}$,

as the Space divided by the Velocity, or as $\frac{\dot{z}}{\sqrt{y}}$,

and the whole Time of describing ADK = F: $\frac{a\dot{z}}{\sqrt{y}}$,

and the Axis AR = F: $b\dot{x}$, also the Curve ADK =

$c\dot{z}$. Then F: $A\dot{z}$ — F: $B\dot{x}$ = F: $\frac{a}{\sqrt{y}} + c \times \dot{z}$ —

$b\dot{x}$. Whence $A = \frac{a}{\sqrt{y}} + c$, $B=b$, and $(A\dot{x}=B\dot{z})$

$+ \frac{a\dot{x}}{\sqrt{y}} = b\dot{z}$, for the Nature of the Curve of swiftest Descent.

If the Velocity at D be supposed to be as y^n , the

equation of the Curve will come out $c\dot{x} + \frac{a\dot{x}}{y^n} = b\dot{z}$.

Ex. 24.

the Length of the Curve ADG, and the Base AG being given; to find the Nature of it, so that its Center of Gravity descends lower than that of any other Line of the same Length, when the Points A, G are fixt.

236.

Let $AB=x$, $BD=y$, $AD=z$; then the Base AG = F: $a\dot{x}$, the Curve ADG = F: $b\dot{z}$, and the Distance of

of the Center of Gravity from AG is $= F: cyz$,
F I G. Prob. XVII. following. Hence $F: A\dot{z} = F: B\dot{x}$
 236. $F: \overline{b+cy} \times \dot{z} = F: a\dot{x}$, and $A = b + cy$, $B = a$,
 ($A\dot{x} = B\dot{z}$) $\overline{b+cy} \times \dot{x} = a\dot{z}$, for the Equation of the
 Curve. In C , where $\dot{x} = \dot{z}$, let $y = d$, and the
 $b + cd = a$. Let $CE = v$, $y = d - v$, and $\overline{b+cd-cv}$
 or $\overline{a-cv} \times \dot{x} = a\dot{z}$, and if $c = -1$, then $\dot{x}$
 $\frac{a\dot{z}}{a+v}$, an Equation to the Catenary, by Ex.
 Prop. XIII.

P R O B. II.

To find the Logarithms of Numbers.

Logarithms are a certain Set of Numbers, so contrived to answer to a Set of Numbers in their natural Order, that the Sum of the Logarithms of any two Numbers shall be the Logarithm of the Product of these Numbers.

Hence therefore, since $1 \times 1 = 1$, the Log. of 1 = Log. 1, that is the Log. 1 = 0.

Also let $AB = C$, then Log. $A + \text{Log. } B = \text{Log. } C$
 and thence Log. A or Log. $\frac{C}{B} = \text{Log. } C - \text{Log. } B$

Again, Log. $A^2 = \text{Log. } A + \text{Log. } A = 2 \text{ Log. } A$
 And Log. $A^3 = \text{Log. } A + \text{Log. } A + \text{Log. } A = 3 \text{ Log. } A$
 And for the same Reason Log. $A^n = n \times \text{Log. } A$.

Lastly, let $AB = 1$, then Log. $A + \text{Log. } B = \text{Log. } 1 = 0$; therefore if the Log. B (or the Log. of a Number greater than 1) be affirmative, the Log. of A (a Number less than 1) is negative.

From what has been said it follows, that if there be a Set of Numbers in geometrical Progression proceeding

ing from 1 both Ways *ad infinitum*; and another F I G. corresponding thereto, in arithmetic Progression proceeding both Ways from 0; then these latter will represent the Logarithms of the former, and both will be express'd in the following Form,

Numbers, $\frac{1}{n^3}, \frac{1}{n^2}, \frac{1}{n}, 1, n, n^2, n^3, n^4, \text{ \&c.}$

Logarithms, $-3l, -2l, -l, 0, l, 2l, 3l, 4l, \text{ \&c.}$

Now let the Number of geometrical Proportionals be increased and their Differences decreased *ad infinitum*, that the Series may contain all possible Numbers; and let the arithmetic Proportionals be in like manner increased; and then the arithmetic Series will contain the Logarithms of all Numbers.

Therefore let n^p be any Number, and pl its Log. the Increment of that Number will be $n^{p+1} - n^p$, this Increment divided by the Number itself, gives

$\frac{n^{p+1} - n^p}{n^p} = n - 1$; likewise the Increment of the

Logarithm is $p+1 \times l - pl$, that is l . Now since $n-1$ and l are the same for any Number and its Logarithm, therefore the Increment of the Logarithm will always be in a given Ratio to the Increment of the Number divided by the Number; therefore if we let $x =$ Number, and $z =$ its Logarithm, and assume the given Quantity M , we shall have $\dot{z} = M \times$

and since the Fluxions are as the Evanescent

Increments, therefore $\dot{z} = M \times \frac{\dot{x}}{x}$, and the Fluent

$= M \times$ Fluent of $\frac{\dot{x}}{x}$. Therefore,

To

FIG. *To find the Logarithm of a Number, divide the Fluxion of the Number proposed, by the Number itself; and find the Fluent, which multiply by the constant Quantity M , and it will give the Logarithm thereof.*

Note, When $M = 1$, the Logarithms are called the Hyperbolic Logarithms; in which Case, the Fluxion of the Logarithm is equal to the Fluxion of the Number divided by the Number. And the Fluxion of the Number is equal to the Number multiplied by the Fluxion of its Logarithm.

EXAMPLE I.

To find the Log. of bx . Here the Fluxion of Log. is $M \frac{b\dot{x}}{bx} = M \frac{\dot{x}}{x}$; but the Fluent of $M \frac{\dot{x}}{x} = \text{Log. } x$, therefore the $\text{Log. } bx = \text{Log. } x$. When $x = 1$, then $\text{Log. } bx = \text{Log. } b$, and $\text{Log. } x = 0$; therefore the Fluent corrected is $\text{Log. } bx - \text{Log. } x = \text{Log. } b$; that is $\text{Log. } bx = \text{Log. } b + \text{Log. } x$.

This finds the Logarithm of a Product having Logarithms of the two Factors given.

Ex. 2.

To find the Logarithm of x^n . Here the Fluxion of its Logarithm $= M \times \frac{nx^{n-1}\dot{x}}{x^n} = M \times \frac{n\dot{x}}{x}$; taking the Fluent, $\text{Log. } x^n = \text{Fluent of } M \times \frac{n\dot{x}}{x} = n \times \text{Log. } x$, which needs no Correction (because when $x = 1$, $x^n = 1$, and $\text{Log. } x = 0$).

This gives a Rule for finding the Logarithm of any Power or Root of a Quantity, when the Logarithm of the Quantity is known.

Ex. 3.

To find the Log. $\frac{n+x}{n}$. The Fluxion of the Log.

$$= M \times \frac{\dot{x}}{n+x} = M \times : \frac{\dot{x}}{n} - \frac{x\dot{x}}{nn} + \frac{x^2\dot{x}}{n^3} -$$

$$\frac{x^3\dot{x}}{n^4} + \mathcal{E}c. \text{ therefore the Log. } \frac{n+x}{n} = M \times : \frac{x}{n} -$$

$$\frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4} + \mathcal{E}c. \text{ If } x \text{ be negative}$$

the Signs of all the odd Powers must be changed.

After the same Manner the Log. $\frac{n}{n-x} = M \times :$

$$+ \frac{x^2}{2n^2} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4} \mathcal{E}c.$$

Otherwise thus.

To find the Log. of $n+x$. Let $n+x=y$; then the

Fluxion of the Log. is $\frac{M\dot{x}}{n+x} = \frac{M\dot{x}}{y}$. And (by

Rule 8, Prop. X.) the Fluent or Log. $n+x :=$

$$M \times : \frac{x}{y} + \frac{x^2}{2y^2} + \frac{x^3}{3y^3} + \frac{x^4}{4y^4} \mathcal{E}c. \text{ and cor-}$$

rected, the Log. $\frac{n+x}{n} - \text{Log. } n = M \times : \frac{x}{y} +$

$$\frac{x^2}{2y^2} + \frac{x^3}{3y^3} \mathcal{E}c. \text{ And Log. } n+x := \text{Log. } n +$$

$$M \times : \frac{x}{y} + \frac{x^2}{2y^2} + \frac{x^3}{3y^3} + \frac{x^4}{4y^4} + \frac{x^5}{5y^5} \mathcal{E}c.$$

$$\text{Also Log. } \frac{n+x}{n} = M \times : \frac{x}{y} + \frac{x^2}{2y^2} + \frac{x^3}{3y^3} +$$

$$\frac{x^4}{4y^4} \mathcal{E}c.$$

C c

COR.

COR. The Log. of $b \times \frac{n+x}{n} = \text{Log. } b + M \times$

$$\frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4} \text{ \&c. And the Log. } b \times \frac{n}{n-x} = \text{Log. } b + M \times : \frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^4}{4n^4} \text{ \&c.}$$

SCHOL. These Series will find the Log. of a Fraction, or the Log. of a Product having the Log. of And it finds the Log. of any large Number $n+x$ having the Log. of b , and making $b=n$.

Ex. 4.

To find the Log. of $\frac{n+x}{n-x}$. Its Fluxion is $= 2M$

$$\frac{n\dot{x}}{nn-xx} = 2M \times : \frac{\dot{x}}{n} + \frac{x^2\dot{x}}{n^3} + \frac{x^4\dot{x}}{n^5} + \frac{x^6\dot{x}}{n^7} + \text{\&c.}$$

+ \&c. Whence the Fluent or Log. $\frac{n+x}{n-x}$

$$2M \times : \frac{x}{n} + \frac{x^3}{3n^3} + \frac{x^5}{5n^5} + \frac{x^7}{7n^7} + \text{\&c.}$$

$$\text{the Log. } \frac{n+x}{n-x} = \frac{2M \times \frac{x}{n}}{1} + \frac{eA}{3} + \frac{eB}{5}$$

$$\frac{eC}{7} + \frac{eD}{9} \text{ \&c. where } e = \frac{xx}{nn}, A, B, C, D$$

are the Numerators of the preceding Terms, with their Signs.

$$\text{COR. The Log. of } b \times \frac{n+x}{n-x} = \text{Log. } b + \frac{2M \times}{1} + \frac{eA}{3} + \frac{eB}{5} + \frac{eC}{7} \text{ \&c.}$$

SCHOL

Schol. The Series in this Example very speedily finds the Log. of a Fraction when x is very small, or very great.

The Series in this Cor. finds the Log. of a Product, having the Log. b one of the Factors given. Likewise it finds the Log. of a large Number $n+x$, having the Log. of b , and making $b = n-x$, and $= 1$ or $\frac{1}{2}$.

Ex. 5.

Find the Log. of n , having the Logarithms of $n-x$ and $n+x$ given.

$$\text{The Fluxion of the Log. of } \frac{n}{\sqrt{nn-xx}} =$$

$$Mx \times \frac{xx}{nn} + \frac{x^3 x}{n^4} + \frac{x^5 x}{n^6} + \frac{x^7 x}{n^8},$$

$$\text{whence the Log. } \frac{n}{\sqrt{nn-xx}} = Mx : \frac{x^2}{2n^2} +$$

$$\frac{x^4}{4n^4} + \frac{x^6}{6n^6} \text{ \&c. but Log. } \frac{n}{\sqrt{nn-xx}} = \text{Log. } n$$

$$\text{Log. } \sqrt{nn-xx} = \text{Log. } n - \frac{\text{Log. } n^2 - x^2}{2} =$$

$$\text{Log. } n - \frac{\text{Log. } n+x}{2} - \frac{\text{Log. } n-x}{2}. \text{ Therefore}$$

$$\text{Log. } n = \frac{\text{Log. } n+x + \text{Log. } n-x}{2} + Mx : \frac{x^2}{2nn}$$

$$+ \frac{x^4}{4n^4} + \frac{x^6}{6n^6} + \text{\&c. or the Log. } n =$$

$$\frac{\text{Log. } n+x + \text{Log. } n-x}{2} + \frac{\frac{1}{2}Me}{1} + \frac{eA}{2} + \frac{eB}{3}$$

$$+ \frac{eC}{4} + \text{\&c. where } e = \frac{xx}{nn}; A, B, C, \text{\&c. the}$$

Numerators of the preceding Terms.

C c 2

SCHOL.

SCHOL. This Series is very proper for finding the Log. of a large prime Number n , having the Log. of the adjoining Numbers $n-1$ and $n+1$ given, and therefore $x = 1$.

Ex. 6.

To find the Log. of n , having the Logarithms of the adjoining Numbers $n-1$ and $n+1$ given.

$$\begin{aligned} \text{Let } 2nn-1 &= y, \text{ then will } \frac{n}{\sqrt{nn-1}} = \frac{\sqrt{\frac{y+1}{2}}}{\sqrt{\frac{y-1}{2}}} \\ &= \sqrt{\frac{y+1}{y-1}}, \text{ the Fluxion of the Log. of } \sqrt{\frac{y+1}{y-1}} \\ &= \frac{-\dot{y}}{yy} - \frac{\dot{y}}{y^4} - \frac{\dot{y}}{y^6} - \frac{\dot{y}}{y^8} \text{ \&c. } \times M \\ \text{therefore the Log. } \sqrt{\frac{y+1}{y-1}} &= \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} \\ &+ \frac{1}{7y^7} \text{ \&c. } \times M; \text{ and when } y \text{ is infinite, Log. } \sqrt{\frac{y+1}{y-1}} \\ &= 0, \text{ and } \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} \text{ \&c. } = 0; \text{ there-} \\ \text{fore still the Log. } \sqrt{\frac{y+1}{y-1}} \text{ or Log. } \frac{n}{\sqrt{nn-1}} & \\ M \times: \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} \text{ \&c. } \text{ When} & \\ \text{Log. } n = \frac{\text{Log. } n+1 \times \text{Log. } n-1}{2} + M \times: \frac{1}{y} & \\ \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \text{\&c.} & \end{aligned}$$

SCHOL. This Series is for the same Purpose, and converges faster than that in Example the 5th.

Ex. 7.

To find the Logarithm of a large Number n , having the Logarithms of $n-x$ and $n+x$ given.

$$\text{Put } d = \text{Log. } n+x - \text{Log. } n-x = \text{Log. } \frac{n+x}{n-x}.$$

$$\text{Then the Log. } n - \text{Log. } n-x = \text{Log. } \frac{n}{n-x} =$$

$$\frac{\text{Log. } \frac{n+x}{n-x} \times \text{Log. } \frac{n}{n-x}}{\text{Log. } \frac{n+x}{n-x}} = d \times \frac{\text{Log. } \frac{n}{n-x}}{\text{Log. } \frac{n+x}{n-x}} =$$

$$\text{By Ex. 3 and 4) } d \times \frac{Mx: \frac{x}{n} + \frac{x^2}{2nn} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4} \mathcal{E}c}{2Mx: \frac{x}{n} + \frac{x^3}{3n^3} + \frac{x^5}{5n^5} + \frac{x^7}{7n^7} \mathcal{E}c}$$

$$= \frac{d}{2} \times : 1 + \frac{x}{2n} + \frac{x^3}{12n^3} + \frac{7x^5}{180n^5} + \mathcal{E}c. \text{ there-}$$

$$\text{fore Log. } n = \text{Log. } n-x + \frac{d}{2} \times : 1 + \frac{x}{2n} + \frac{x^3}{12n^3} + \frac{7x^5}{180n^5} + \mathcal{E}c. \text{ Or thus, since } \frac{d}{2} =$$

$$\frac{\text{Log. } n+x - \text{Log. } n-x}{2}, \text{ therefore Log. } n = \frac{\text{Log. } n+x + \text{Log. } n-x}{2} + \frac{d}{2} \times : \frac{x}{2n} + \frac{x^3}{12n^3}$$

$$+ \frac{7x^5}{180n^5} + \mathcal{E}c.$$

SCHOL. This Series converges far faster than either of the former Examples.

Ex. 8.

To find the Hyperbolic Logarithm of 10.

$$\text{Since } \frac{5^{10} \times 2^{30}}{4^{10} \times 10^9} \text{ or } \frac{5^{10}}{4^{10}} \times \frac{1024^3}{1000^3} = 10; \text{ therefore } 10 \text{ Log.}$$

$$10 \text{ Log. } \frac{5}{4} + 3 \text{ Log. } \frac{1024}{1000} = \text{Log. } 10. \text{ By Ex. } 1$$

$$\text{Log. } \frac{5}{4} = 2M \times : \frac{1}{9} + \frac{1}{3} \times \frac{1}{9^2} + \frac{1}{5} \times \frac{1}{9^3} \text{ \&c. therefore}$$

$$10 \text{ Log. } \frac{5}{4} = \frac{20}{9} + \frac{Ae}{3} + \frac{Be}{5} + \frac{Ce}{7} + \frac{De}{9} + \text{\&c.}$$

$$(\text{putting } e = \frac{1}{81}). \text{ Likewise } 3 \text{ Log. } \frac{1024}{1000} = \frac{18}{253}$$

$$\frac{Ae}{3} + \frac{Be}{5} + \frac{Ce}{7} \text{ \&c. (putting } e = \frac{9}{64009})$$

Here $M=1$, A = first Term of each Series, and $C, D, \text{\&c.}$ the Numerators of each preceding Term.

The particular Terms of each Series being found will be as follows.

1) A	=	2,22222222222222222222222222222222
3) Ae	=	91449474165523548239598
5) Be	=	677403512337211468441
7) Ce	=	5973575946536256335
9) De	=	57359439815848826
11) Ee	=	579388280968170
13) Fe	=	6052489164910
15) Ge	=	64759143328
17) He	=	705437291
19) Ie	=	7792354
21) Ke	=	87040
23) Le	=	981
25) Me	=	11

$$10 \text{ Log. } \frac{5}{4} = 2,2314355131420975576629507$$

A	=	,0711462450592885375494071
3) Ae	=	33345113214995643214
5) Be	=	2813098335561819
7) Ce	=	282525950815
9) De	=	30896931
11) Ee	=	3554

$$3 \text{ Log. } \frac{1024}{1000} = ,0711495798519481263550404$$

$$\text{H. Log. } 10 = 2.3025850929940456840179911 \text{ \&c.}$$

COROL

COROL. Hence therefore the Number M , mentioned before, and made Use of in the foregoing Examples, will be known for the common Logarithms. For since the common Logarithm of 10 is 1, therefore

$$302585 \text{ \&c. } \times M = 1. \text{ Whence } M = \frac{1}{2.302585 \text{ \&c. }}$$

$$,4342944819032518276511289 \text{ \&c.}$$

After the same Manner may *Brig's* or the common Logarithms be found, since we know the Number M . or let the former Series or $10 \text{ Log. } \frac{1}{2} = P$; the

$$\text{other Series or } \text{Log. } \frac{2^{30}}{10^9} = Q; \text{ then the } \text{Log. } 2 =$$

$$\frac{10 - MP}{30} \text{ or } \frac{9 + MQ}{30}, \text{ whence are had the Logarithms}$$

$$\text{4 and 8; likewise the } \text{Log. } 5 = \frac{20 + MP}{30}. \text{ And}$$

thus,

$$\text{The Log. of } 7 = \text{Log. } 8 \times \frac{7}{8} = \text{Log. } 8 - \frac{\frac{1}{8}M}{1}$$

$$- \frac{\frac{1}{80}A}{3} + \frac{\frac{1}{80}B}{5} + \frac{\frac{1}{80}C}{7} + \frac{\frac{1}{80}D}{9} \text{ \&c. by Example the 4th. And likewise}$$

$$\text{Log. } 3 = \text{Log. } 2 \times \frac{3}{2} = \text{Log. } 2 + \frac{\frac{1}{2}M}{1} + \frac{\frac{1}{20}A}{3} - \frac{\frac{1}{20}B}{5} + \frac{\frac{1}{20}C}{7} \text{ \&c. from whence is had the Log. 6 and 9.}$$

$$\text{Or the } \text{Log. } 9 = \text{Log. } 10 \times \frac{9}{10} = 1 - \frac{\frac{1}{10}M}{1} + \frac{\frac{1}{10}A}{2} + \frac{\frac{1}{10}B}{3} + \frac{\frac{1}{10}C}{4} \text{ \&c. and } \text{Log. } 11 = \text{Log. } 10$$

$$\frac{11}{10} = 1 + \frac{\frac{1}{10}M}{1} - \frac{\frac{1}{10}A}{2} - \frac{\frac{1}{10}B}{3} - \frac{\frac{1}{10}C}{4} - \text{ \&c.}$$

both these by Example the 3d. And so for others.

SCHOLIUM I.

The Rules in Example the 3d and 4th are the two Methods by which the Logarithm of a Number can be found without the Help of other given Logarithms. And therefore to find the Log. of a given Number thereby, we must take two or more such fractional Numbers, that the Product of some Powers of these Numbers may make the Number whose Logarithm is sought: And then there will be two or more Series, which will give the required Logarithm; in the same Manner as is shewn in Ex. for the Log. of the Number 10.

SCHOLIUM 2.

To find the Number from the given Logarithm.

Let n = any Number,

l = Log. of it,

$n+x$ = any other Number,

$l+z$ = its Log. = L .

$$m = \frac{1}{M} = 2.302585 \text{ \&c. for the common Logarithm}$$

Then we shall have $\dot{z} = \frac{M\dot{x}}{n+x}$, or $M\dot{x} = n\dot{z} + x\dot{z}$ that is $\dot{x} = mn\dot{z} + mx\dot{z}$.

To find the Fluent, by Sir I. Newton's Method (Prop. X. Rule 2.)

	$mn\dot{z}$	*	*	*
$+ mx\dot{z}$	$+ m^2nz\dot{z}$	$+ \frac{m^3nz^2\dot{z}}{2}$	$+ \frac{m^4nz^3\dot{z}}{2.3}$	
$\dot{x} = mn\dot{z}$	$+ m^2nz\dot{z}$	$+ \frac{m^3nz^2\dot{z}}{2}$	$+ \frac{m^4nz^3\dot{z}}{2.3}$	
$x = mnz$	$+ \frac{m^2nz^2}{2}$	$+ \frac{m^3nz^3}{2.3}$	$+ \frac{m^4nz^4}{2.3.4}$	&c.

Which

ect. II. of FLUXIONS.

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Which is correct; because when $x = 0$, $z = 0$. F I G.

Therefore $n + x = n \times 1 + mz + \frac{m^2 z^2}{2} + \frac{m^3 z^3}{2.3} +$

$\frac{m^4 z^4}{3.4} + \frac{m^5 z^5}{2.3.4.5}$ &c. that is $= n \times 1 + mz +$

$\frac{mz}{2}A + \frac{mz}{3}B + \frac{mz}{4}C + \frac{mz}{5}D$ &c. = Num-

ber whose Log. is $l + z$, where $z = L - l$.

COR. If $n = 1$, then $l = 0$, and $z = L$. Therefore

$+ mL + \frac{mL}{2}A + \frac{mL}{3}B + \frac{mL}{4}C + \frac{mL}{5}D$

&c. = Number whose Log. is L .

And if $m = 1$, then

$+ L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5}$ &c. =

Number whose Hyperbolic Logarithm is L .

P R O B. III.

To draw Tangents to Curves.

1. In all geometrical Curves, where there is given 25.

the Relation of the Abscissa and Ordinate. Let AP

$= x$, $PM = y$, and draw mp parallel and infinitely

near to MP , and MR parallel to AP , and draw the

Tangent MT , and let Pp or $MR = x'$, $Rm = y'$. Then

by the similar Triangles TPM and MRm , $y' : x' :: y :$

$x = PT$ the Subtangent, that is $\frac{y x'}{y} = PT$.

D d

Therefore

FIG. Therefore, by Help of the Equation of the Curve,

25. terminate $\dot{x}$ or $\dot{y}$ out of the Quantity $\frac{y\dot{x}}{\dot{y}}$, and you get the Value of the Subtangent.

2. In mechanical or transcendent Curves referred to Axis, the Subtangent $\frac{y\dot{x}}{\dot{y}}$ may be clear'd of the Fluxion by the Equation of the Curve, by the Help of those the known Curve it is related to.

26. 3. And in any Curve QM referred to a fixt Point, draw AT perpendicular to AM , and let $AM=y=A$

$Rn=\dot{y}$, $RM=\dot{x}$; and by similar Triangles $\frac{y\dot{x}}{\dot{y}}$ or

$=AT$ the Subtangent, out of which the Fluxions may be exterminated as before.

There are several other Methods of drawing Tangents, but this is the most general and easy.

EXAMPLE I.

27. To draw a Tangent to the Circle. Let $AP=PM=y$, Radius $AC=r$; then will $2rx-xx=$
whence $2r\dot{x}-2x\dot{x}=2y\dot{y}$, and $\dot{x} = \frac{y\dot{y}}{r-x}$. Therefore $PT = \frac{y\dot{x}}{\dot{y}} = \frac{yy}{r-x} = \frac{2rx-xx}{r-x}$.

Ex. 2.

28. To draw a Tangent to the Ellipsis. Let Transverse $AB=2a$, Latus rectum $=b$, $AP=x$, $PM=y$. Then by the Nature of the Curve $\frac{2ayy}{b} = 2ax-xx$, and in Fluxions $\frac{4ay\dot{y}}{b} = 2a\dot{x} - 2x\dot{x}$, or $\dot{y} = \frac{ab\dot{x}-bx\dot{x}}{2ay}$

consequently $PT = \frac{y\dot{x}}{\dot{y}} = \frac{2ayy}{ba-bx} = \frac{2ax-xx}{a-x}$. FIG.

Ex. 3.

To draw Tangents to all sorts of Parabolas. Let $AP=x$, $PM=y$, and $x=y^m$, putting the Parameter

25. Then $\dot{x}=my^{m-1}\dot{y}$, therefore $PT = \frac{y\dot{x}}{\dot{y}} = my^m$

26. And therefore in the common Parabola, where $m=2$, $TP=2x=2AP$.

Ex. 4.

To draw a Tangent to an Hyperbola. Let $AP=x$, $PM=y$, Transverse $=2a$, Latus Rectum $=b$. Then

25. The Property of the Figure is, $\frac{2ayy}{b} = 2ax+xx$; in

Fluxions $\frac{4ay\dot{y}}{b} = 2a\dot{x} + 2x\dot{x}$ or $\dot{y} = \frac{a\dot{x}+x\dot{x}}{2ay} b$;

therefore $PT = \frac{y\dot{x}}{\dot{y}} = \frac{2ayy}{ba+bx} = \frac{2ax+xx}{a+x}$.

Ex. 5.

Let MB be an Hyperbola between the Affymptotes; 29. Let $AP=x$, $PM=y$, and $aa=xy$; then $x\dot{y}+y\dot{x}=0$,

$\dot{x} = -\frac{xy\dot{y}}{y}$; whence $PT = \frac{y\dot{x}}{\dot{y}} = -x$, and the

Negative Sign shews that T lies on the contrary Side of PM with A .

Ex. 6.

To draw a Tangent to the Cissoid AM . Let AP 30. $=x$, $PM=y$, $AB=a$, BC the Affymptote. Then

by the Nature of the Curve $ay^2-xy^2=x^3$; whence $ay\dot{y}-2xy\dot{y}-y^2\dot{x}=3x^2\dot{x}$, or $\dot{x} = \frac{2ay-2xy}{yy+3xx}\dot{y}$,

D d 2

whence

FIG. whence $PT = \frac{y\dot{x}}{\dot{y}} = \frac{2ayy - 2xyy}{y^2 + 3x^2}$, or expunging

30.

$$yy, PT = \frac{2ax - 2xx}{3a - 2x}.$$

Ex. 7.

31.

To draw a Tangent to the Conchoid. Let $CA = a$, $AV = a$, $AP = x$, $PM = y$. By the Nature of the Curve, $b + x \sqrt{aa - xx} = xy$, whence $\dot{x} \sqrt{aa - xx} -$

$$- \frac{\dot{x}x \times b + x}{\sqrt{aa - xx}} = xy + y\dot{x}; \text{ whence } PT = \frac{y\dot{x}}{\dot{y}} =$$

$$\frac{yx \sqrt{aa - xx}}{aa - 2xx - bx - y \sqrt{aa - xx}} = \frac{b + x \times aax - x^3}{-baa - x^3}$$

and being negative, it lies on the contrary Side of PM

Otherwise thus;

32.

Let $AP = v$, $AM = y$, $PM = a$, $AH = b$, $MR =$. Then by the Nature of the Curve $v + a = y$, and

$$\dot{v} = \dot{y}; \text{ and by similar Triangles } Pp = \frac{v\dot{x}}{y}, \text{ and}$$

$$\dot{v} = \frac{v \times Pp}{b} = \frac{v^2 \dot{x}}{by}, \text{ and thence } \dot{x} = \frac{by\dot{y}}{vv} \text{ or } \dot{x} =$$

$$\frac{by\dot{y}}{vv}; \text{ therefore } AT = \frac{y\dot{x}}{\dot{y}} = \frac{byy}{vv}.$$

Ex. 8.

25.

Let AM be the Catenary, $AP = x$, $PM = y$, $AM = z$ by the Nature of the Curve $zz = 2ax + xx$, whence $z\dot{z}$

$$= a\dot{x} + x\dot{x}, \text{ and } \dot{z} = \frac{a + x}{z} \dot{x}; \text{ but } \dot{y}^2 = \dot{z}^2 - \dot{x}^2 =$$

$$\frac{a + x}{z}^2 \dot{x}^2 - \dot{x}^2 = \frac{a + x}{z}^2 - z\dot{z} \dot{x}^2 = (\text{expunging } z\dot{z})$$

$\frac{aa\dot{x}^2}{zz}$, and $\dot{y} = \frac{a\dot{x}}{z}$; therefore $PT = \frac{y\dot{x}}{\dot{y}}$ F I G. 25.
 $\frac{zy}{a}$

Ex. 9.

To draw a Tangent to the Cycloid. Let $AP = x$, $33.$
 $PM = y$, $AC = 2a$, $PB = u$, Arch $AB = z$. By the
 Nature of the Figure $y = z + u$, and thence $\dot{y} = \dot{z} + \dot{u}$,
 from the Circle $\dot{z} = \frac{a}{u}\dot{x}$, and $\dot{u} = \frac{a-x}{u}\dot{x}$; whence
 $PT = \left(\frac{y\dot{x}}{\dot{z} + \dot{u}} = \right) \frac{uy}{2a-x} = \frac{AP}{PB}y$.

Ex. 10.

To draw a Tangent to the Quadratrix AMB . Let $34.$
 $AE = b$, $CB = r$, $AP = x$, $PM = y$, $DN = z$, $NI = s$,
 $AM = v$, $NQ = t$, $RM = u$. Now by the Nature of the
 Curve $bz = ry$, therefore $b\dot{z} = r\dot{y}$, and by the Property
 of the Circle $rs = t\dot{z}$; whence $\dot{z} = \frac{r\dot{y}}{b} = \frac{rs}{t}$, and
 $\dot{y} = \frac{ty}{b}$. By similar Triangles $ty = su$, and thence
 $ty = su$, that is (because $u = b - x$ or
 $\dot{u} = -\dot{x}$, and $rr - ss = tt$ or $t = \frac{-s\dot{s}}{\dot{t}}$) $ty =$
 $\dot{y} = (us - s\dot{x}) = \frac{tuy}{b} - s\dot{x}$; whence $\dot{x} = \frac{tuy}{bs}$
 $\frac{y\dot{y}}{b} - \frac{t\dot{y}}{s}$. Therefore $PT = \frac{y\dot{x}}{\dot{y}} = \frac{tuy}{bs} +$
 $-\frac{ty}{s} = \left(\text{because } u = \frac{ty}{s} \right) \frac{u^2 + y^2}{b} - u =$
 $-\frac{ty}{s} - u$.

Ex.

EX. II.

- FIG. Let AS be any given Curve, and AM another Curve
 35. such that $AP = \text{Arch } AS$, and Ordinate $PM = AQ$, to draw the Tangent TM .

Let $AP = x$, $PM = y$, $AQ = v$, Tangent $SR = s$, $Q = t$, $AS = z$. By the Nature of the Figure $x = z$, and $v = y$, therefore $\dot{x} = \dot{z}$ and $\dot{v} = \dot{y}$; and by similar Triangles $\dot{v} = \frac{QR}{SR} \dot{z} = \frac{t\dot{x}}{s}$; therefore $PT = \left(\frac{y\dot{x}}{\dot{v}} \right) = \frac{sy}{t}$.

EX. 12.

36. Let BM be an exponential Curve, $AP = x$, $PM = a^x = y$. Let A, Y be the hyperbolic Logarithms of a, y ; then $xA = Y$, and $A\dot{x} = \dot{Y} = \frac{\dot{y}}{y}$; whence $PT = \frac{y\dot{x}}{\dot{y}} = \frac{1}{A}$ an invariable Quantity: Therefore BM is the Logarithmic Curve.

EX. 13.

36. Let the Nature of an exponential Curve be expressed by this Equation $x^x = y$; Let X, Y be the hyperbolic Logarithms of x, y ; then $xX = Y$, and $x\dot{X} + X\dot{x} = \dot{Y}$ but $\dot{X} = \frac{\dot{x}}{x}$, and $\dot{Y} = \frac{\dot{y}}{y}$, (by Prob. II.) therefore $\dot{x} + X\dot{x} = \frac{\dot{y}}{y}$, or $y\dot{x} + yX\dot{x} = \dot{y}$, whence $PT = \frac{y\dot{x}}{y\dot{x} + yX\dot{x}} = \frac{1}{1 + X}$.

EX. 14.

37. To draw a Tangent to Archimedes's Spiral. Let Radius $AP = r$, Arch $QP = z$, $AM = y$, c a given Line

by the Nature of the Figure $rz = cy$, and thence F I G.
 $= \frac{cy}{r}$, and by similar Triangles $\dot{x} = \frac{y\dot{z}}{r} = \frac{cy}{rr}\dot{y}$. 37.

hence $AT = \frac{y\dot{x}}{\dot{y}} = \frac{cy}{rr}$.

Ex. 15.

To draw a Tangent to the reciprocal Spiral. Let 38.
 radius $BA = a$, Arch $BC = b$, $BP = z$, $AM = y$,
 $BP = z$. The Equation of the Curve is $ab = zy$, and
 therefore $zy + y\dot{z} = 0$; and by similar Triangles $y\dot{z} =$
 $-a\dot{x}$, or $y\dot{z} = -a\dot{x}$, whence $zy - a\dot{x} = 0$, or $\dot{x} =$
 $\frac{zy}{a}$. Therefore $AT = \frac{y\dot{x}}{\dot{y}} = \frac{zy}{a} = b$, a given
 quantity.

Ex. 16.

Let AMD be a Spiral of such a kind, that putting 39.
 AM or $AC = y$, Arch $MC = v$, $AD = a$, a given Line,
 and $av^2 = y^3$. Let Arch $BE = z$, and by the similar
 sectors AMC , AEB , $av = yz$, whence expunging v ,
 $a = ay$, also rM or $\dot{x} = \frac{y\dot{z}}{a}$, or $\dot{x} = \frac{y\dot{z}}{a}$. And
 $zy\dot{z} = ay\dot{y}$; whence $\dot{x} = \frac{y\dot{y}}{2z} = \frac{zy}{2a}$, and therefore

$$AT = \left(\frac{y\dot{x}}{\dot{y}} = \frac{yz}{2a} = \frac{av}{2a} = \right) \frac{1}{2}v.$$

PROB.

P R O B. IV.

To find the Point of Inflexion or contrary Flexure of a given Curve.

F I G. The Point of Inflexion or contrary Flexure is the Point which separates the convex from the concave Part of the Curve.

40. In the Curve AM any way related to the Axis A let $AP = x$, Ordinate $PM = y$; draw the Ordinate pm parallel and infinitely near PM , and Mr parallel to AP . Now the Ratio of Mr to rm is the greatest or least possible in the Point of contrary Flexure that is $\frac{Mr}{mr}$ or $\frac{\dot{x}}{\dot{y}}$ is a Maximum or Minimum

whence (by Prob. I.) its Fluxion $\frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^2} = 0$, or $-\dot{x}\ddot{y} = 0$; hence if $\ddot{x} = 0$, then $\ddot{y} = 0$, or if $\ddot{y} = 0$ then $\ddot{x} = 0$.

41. In a Curve BM referred to the fixed Point P , take Mm infinitely small, and draw Pm , and Mr perpendicular to it, draw the Tangent MT and P perpendicular to it; and let $PM = y$, Curve $BM =$

$Mr = \dot{x}$, $mr = \dot{y}$. Then, by similar Triangles

$$PT = \frac{y\dot{x}}{\dot{z}} = \frac{y\ddot{x}}{\ddot{z}}. \text{ But in the Point of Inflexion}$$

the Perpendicular PT is a Maximum or Minimum therefore its Fluxion or the Fluxion of $\frac{y\dot{x}}{\dot{z}} = 0$, that

$$\text{is } \frac{y\dot{z}\ddot{x} + \dot{x}\dot{y}\ddot{z} - y\dot{x}\ddot{z}}{\dot{z}^2} = 0, \text{ or } y\dot{z}\ddot{x} - y\dot{x}\ddot{z} + \dot{x}\dot{y}\ddot{z} = 0$$

out of which any of the Fluxions may be exterminated by Help of the Equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$ and its Fluxion, making any of the Fluxions invariable, or the second Fluxion = 0. Hence therefore to find the Point of Inflexion, the Rule is

1. In Curves referred to an Axis, let $AP = x$, $PM = y$, 40.
Put the Equation of the Curve into Fluxions, and again the resulting Equation into Fluxions, making both $\dot{x}$ and $\dot{y}$ given Quantities.

2. In Curves related to a Pole or given Point P; let 41.
 $PM = y$, $BM = z$, and Mm , mr , Mr will be as $\dot{y}$, $\dot{x}$. Put the Equation of the Curve into Fluxions, and the resulting Equation into Fluxions again, writing for some first Fluxion each Operation: Then you will determine the other Fluxion which must be substituted into one of the following Equations that contains that other Fluxion. And then you will get the Value of y .

$$\text{If } \dot{x} = 1. \text{ then } y = \frac{1 + \dot{y}^2}{\ddot{y}} = \frac{\dot{z}\sqrt{\dot{z}^2 - 1}}{\ddot{z}}.$$

$$\text{If } \dot{y} = 1. \quad y = \frac{\dot{x} + \dot{x}^3}{-\ddot{x}} = \frac{\dot{z}^3 - \dot{z}}{-\ddot{z}}.$$

$$\text{If } \dot{z} = 1. \quad y = \frac{1 - \dot{y}^2}{\ddot{y}} = \frac{\dot{x}\sqrt{1 - \dot{x}^2}}{-\ddot{x}}.$$

EXAMPLE I.

Let the Equation of the Curve be $ax^2 = x^2y + aay$.

This in Fluxions is $2ax\dot{x} = 2yx\dot{x} + x^2\dot{y} + a\dot{a}\dot{y}$; which

put again into Fluxions, $2a\dot{x}^2 = 2y\dot{x}^2 + 2x\dot{x}\dot{y} + 2x\dot{x}\dot{y}$

$= 2y\dot{x}^2 + 4x\dot{x}\dot{y}$, and expunging $\dot{y}$, $a\dot{x}^2 = y\dot{x}^2 + 2x\dot{x}\dot{x}$

$\frac{2ax - 2yx}{aa + xx} \dot{x}$; that is by Reduction $aa + xx = 4xx$;

hence $x = a\sqrt{\frac{1}{3}}$.

E c

Ex.

Ex. 2.

FIG. Let *BM* be *Nichomedes's Concoïd*. $AB=a$, $AC=$

42. then its Nature is $yx = b + x\sqrt{aa-xx}$. In Fluxions

$$y\dot{x} + x\dot{y} = \dot{x}\sqrt{aa-xx} - \frac{bx\dot{x} + x^2\dot{x}}{\sqrt{aa-xx}}; \text{ and expun}$$

$$\text{ing } y, \frac{-x^3 - aab}{xx\sqrt{aa-xx}}\dot{x} = \dot{y}, \text{ and this again in Fluxions}$$

$$\text{is } \frac{2a^4b - a^2x^3 - 3aabxx}{aax^3 - x^5\sqrt{aa-xx}}\dot{x}^2 = 0, \text{ and reduced } x^3$$

$$3bxx = 2aab.$$

Ex. 3.

43. Let *AFK* be a *Cycloid* of such a Kind, that Arch *AD*
 $DM :: ADB : BK$. Let $AB=2r$, $ADB=a$, $BK=$

$$PD=z, AD=u; \text{ then } y = z + \frac{bu}{a}, \text{ and } \dot{y} = \dot{z} +$$

$$\frac{b\dot{u}}{a}; \text{ by the Property of the Circle } \dot{z} = \frac{r-x}{\sqrt{2rx-xx}}$$

$$\text{and } \dot{u} = \frac{r\dot{x}}{\sqrt{2rx-xx}}, \text{ whence } \dot{y} = \frac{ar-ax+rb}{a\sqrt{2rx-xx}}$$

$$\text{in Fluxions } \frac{brx-arr-brr}{2rx-xx}^{\frac{1}{2}}\dot{x}^2 = 0, \text{ reduced } x^{\frac{1}{2}}$$

$$r + \frac{ar}{b}.$$

Ex. 4.

44. Let *BM* be the *belicoid Parabola*, Radius *PB*
 $PE=r$, Arch $BE=v$, $PM=y$. Then by the Na-
 ture of the Curve $av = r-y$, whence $a\dot{v} = -$

$$x\dot{r}-y, \text{ but by similar Sectors } \dot{v} = \frac{r\dot{x}}{y}, \text{ or } \dot{v} = \frac{r\dot{x}}{y}$$

when

hence $rax = -2yy \times \overline{r-y}$; and $\dot{y} = \frac{ar}{2y \times \overline{y-r}}$

putting $\dot{x} = 1$, and $\ddot{y} = \frac{-ary}{2yy} \times \frac{2y-r}{y-r}$, whence

$$= \frac{1+\dot{y}^2}{\ddot{y}} = \frac{4yy \times \overline{y-r}^2 + aarr}{-2ary \times 2y-r}, \text{ and ex-}$$

expunging $\dot{y}$, $4y^2 \times \overline{y-r}^2 + aarr = 2ary \times \overline{2y-r} \times \overline{-ar}$, that is $4yy \times \overline{y-r}^3 + aarr \times \overline{y-r} + aarr$

$\times 2y-r = 0$. Which reduced gives $4y^5 - 12ry^4 + 2r^2y^3 - 4r^3y^2 + 3r^2a^2y - 2r^3a^2 = 0$.

Ex. 5.

Suppose BM to be a sort of Spiral, PT or $PR=a$, 45.
Arch $RT=v$, and let $a^3=vyv$ express its Nature; then

$vyy+yv\dot{v}=0$, but by similar Triangles $\dot{v} = \frac{a\dot{x}}{y}$,

whence $2vy\dot{y}+ay\dot{x}=0$, and $\dot{x} = \frac{-2v}{a}$ putting

$\dot{y}=1$; and expunging v , $\dot{x} = \frac{-2aa}{yy}$, and $\ddot{x} =$

$\frac{4aa}{y^3}$; therefore $y = \frac{\dot{x}+\dot{x}^3}{-\ddot{x}} = \frac{1}{2}y + \frac{2a^4}{y^3}$, or y^4

$= 4a^4$, whence $y = a\sqrt{2}$.

PROB. V.

To find the Radius of Curvature in Curves.

Let Amq be a Curve referred to the Axis AE , and 46.
suppose C to be the Center, and Cm the Radius of
Curvature in any Point m , and describe the equicurve

E e 2

Circle

FIG. Circle Dm coinciding with the Curve in mn . D
 46. DC , mr parallel, and AF perpendicular to AB ,
 produce mB to H , and draw nb parallel and infinitely near mBH ; and call the Abscissa AB , x ; perpendicular Ordinate Bm , y ; Am , z ; mC , r ; DF , AF or BH , c ; mr , $\dot{x}$; rn , $\dot{y}$; and mn , $\dot{z}$; then $CH = r - b - x$.

By the Nature of the Circle $DH \times 2DC = DHm^2$, that is $2rx + 2rb - bb - 2bx - xx = cc + 2cy + y^2$; which put into Fluxions $r\dot{x} - b\dot{x} - x\dot{x} = c\dot{y} + y\dot{y}$; which put into Fluxions again, $r\ddot{x} - b\ddot{x} - x\ddot{x} - \dot{x}^2 = c\ddot{y} + y\ddot{y}$; therefore $r - b - x \times \ddot{x} - \dot{y} \times c + y = \dot{x}^2 + \dot{y}^2 = \dot{z}^2$.

by similar Triangles $r - b - x = \frac{r\dot{y}}{\dot{z}} = \frac{r\dot{y}}{\dot{z}}$, and
 $= \frac{r\dot{x}}{\dot{z}} = \frac{r\dot{x}}{\dot{z}}$, whence the former Equation becometh

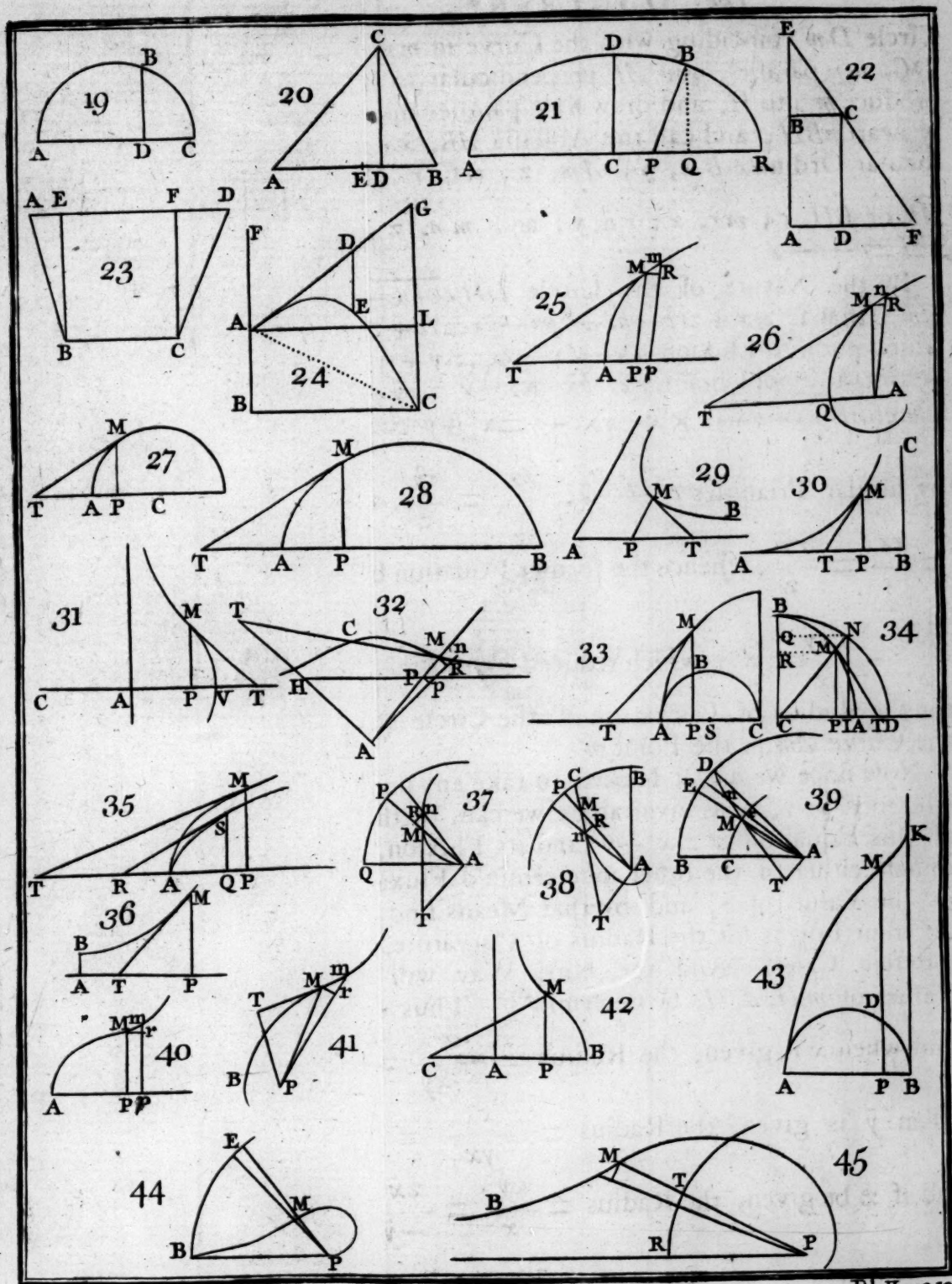
$\frac{r\dot{y}\ddot{x}}{\dot{z}} - \frac{r\dot{x}\ddot{y}}{\dot{z}} = \dot{z}^2$; which reduced gives $r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$
 for the Radius of Curvature of the Circle Dm of the Curve Am in the Point m .

Now since we are at Liberty to take any one of Fluxions $\dot{x}$, $\dot{y}$, $\dot{z}$, as invariable; we can, by the Help of this Equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$ and its Fluxion, eliminate either of the other indetermin'd Fluxions of the Value of r , and by that Means find several different Forms for the Radius of Curvature, to different Cases. And the same Way will serve for the Values of mH , CH , be determin'd. Thus we find

when $\dot{x}$ is given, the Radius $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}} = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$

when $\dot{y}$ is given, the Radius $= \frac{\dot{z}^3}{\dot{y}\ddot{x}} = \frac{\dot{z}^3}{\dot{y}\ddot{x}}$

and if $\dot{z}$ be given, the Radius $= \frac{\dot{z}\dot{y}}{\ddot{x}} = \frac{\dot{z}\dot{x}}{-\ddot{y}}$. Ag



Again in Spirals or Curves referr'd to a fixt Point *B*,
AMn. Let *C* be the Center, and *CM* the Radius
 Curvature in any Point *M*; and describe the Circle
Mn coinciding with the Curve in the infinitely small
 part *Mn*. Draw *Bn* infinitely near *BM*, and *CD*,
 perpendicular to *BM*; and call *BM*, *y*; Curve

M, *z*; Radius *CM*, *r*; *DE*, *v*; *Ee*, $\dot{v}$; *Mr*, $\dot{x}$;
En, $\dot{z}$; *rn*, $\dot{y}$. By the similar Triangles *MEC* and

$$Mr, ME = \frac{r\dot{x}}{\dot{z}} = \frac{r\dot{x}}{\dot{z}}, \text{ and } EC = \frac{r\dot{y}}{\dot{z}} = \frac{r\dot{y}}{\dot{z}},$$

$$\text{Therefore } v = r - \frac{r\dot{y}}{\dot{z}}, \text{ and thence } \dot{v} = \frac{r\dot{y}\ddot{z} - r\dot{z}\ddot{y}}{\dot{z}^2}:$$

and by the similar Triangles *BEe*, *BMr*, it is *BE* :
BM :: *Ee* : *Mr*, that is $y - \frac{r\dot{x}}{\dot{z}} : y :: (\dot{v} : \dot{x} :: \dot{v}$

$$::) \frac{r\dot{y}\ddot{z} - r\dot{z}\ddot{y}}{\dot{z}^2} : \dot{x}, \text{ which multiply'd and re-}$$

duced gives $r = \frac{y\dot{x}\dot{z}^2}{\dot{z}\dot{x}^2 + y\dot{y}\ddot{z} - y\dot{z}\ddot{y}}$, for the Radius of
 Curvature of the Circle *DM*, or of the Curve *AM*

M. Hence also $ME = \frac{y\dot{x}^2\dot{z}}{\dot{z}\dot{x}^2 + y\dot{y}\ddot{z} - y\dot{z}\ddot{y}}$, and

$$C = \frac{y\dot{y}\dot{x}\dot{z}}{\dot{z}\dot{x}^2 + y\dot{y}\ddot{z} - y\dot{z}\ddot{y}}.$$

Now since we can make any one of the Fluxions
 $\dot{y}$, $\dot{z}$, invariable; we can expunge either of the
 it and its Fluxion by the Help of this Equation
 $\dot{x}^2 + \dot{y}^2$ and its Fluxion, and thence obtain Variety
 different Forms for each of the Quantities *CM*, *ME*,
C, as before.

Otherwise thus, when we have the Perpendicular
 on the Tangent drawn to any Point *M* of the Curve,
 we can find the Radius of Curvature *CM* very easily
 thus;

F I G. thus; let BP , Bp be perpendicular to the Tangent

48. MP , np , and let $BP = u$, $pq = \dot{u}$, the rest as before

Then by the similar Triangles pMq , MCn ; and

Mrn , MPB ; $r\dot{u} = pM \times Mn = \dot{y}y$, or $r\dot{u} = \dot{y}y$

whence $r = \frac{\dot{y}y}{\dot{u}}$. Hence therefore to find the Radius

of Curvature the Rules are,

46. 1. In Curves whose Ordinates are perpendicular to the Abscissa, for the Curve, Abscissa, and Ordinate, put z , x , y : Put the Equation of the Curve (reduced as low and simple as possible) into Fluxions, writing 1 for any one of the first Fluxions contained in it; then put the resulting Equation into Fluxions again, writing 1 for the same first Fluxion. From the first Equation the other Fluxion will be determin'd, and by the second you will get the second Fluxion; which being had, substitute them into one of the following Forms that contains the first and second Fluxion.

F O R M S.

$$\begin{aligned} \text{If } \dot{x} = 1, \left\{ \begin{aligned} \text{Radius } mC &= \frac{(1 + \dot{y}^2)^{\frac{3}{2}}}{-\ddot{y}} = \frac{\dot{z}^2 \sqrt{z^2 - 1}}{-\ddot{z}} \\ mH &= \frac{1 + \dot{y}^2}{-\ddot{y}} = \frac{\dot{z}^2 \sqrt{z^2 - 1}}{-\ddot{z}} \\ CH &= \frac{\dot{y} \times 1 + \dot{y}^2}{-\ddot{y}} = \frac{\dot{z} \times \dot{z}^2 - 1}{-\ddot{z}} \end{aligned} \right. \\ \\ \text{If } \dot{y} = 1, \left\{ \begin{aligned} \text{Radius } mC &= \frac{(1 + \dot{x}^2)^{\frac{3}{2}}}{\ddot{x}} = \frac{\dot{z}^2 \sqrt{z^2 - 1}}{\ddot{z}} \\ mH &= \frac{\dot{x} \times 1 + \dot{x}^2}{\ddot{x}} = \frac{\dot{z} \times \dot{z}^2 - 1}{\ddot{z}} \\ CH &= \frac{1 + \dot{x}^2}{\ddot{x}} = \frac{\dot{z}^2 \sqrt{z^2 - 1}}{\ddot{z}} \end{aligned} \right. \end{aligned}$$

$$\text{If } \dot{z} = 1, \left\{ \begin{array}{l} \text{Radius } mC = \frac{\sqrt{1-\dot{x}^2}}{\ddot{x}} = \frac{\sqrt{1-\dot{y}^2}}{-\ddot{y}} \\ mH = \frac{\dot{x}\sqrt{1-\dot{x}^2}}{\ddot{x}} = \frac{1-\dot{y}^2}{-\ddot{y}} \\ CH = \frac{1-\dot{x}^2}{\ddot{x}} = \frac{\dot{y}\sqrt{1-\dot{y}^2}}{-\ddot{y}} \end{array} \right.$$

But in the Vertices of Curves, where they cut the Abscissa at right Angles, take $\frac{y\ddot{y}}{\dot{x}}$ for the Radius of Curvature; and in the highest Point, where they cut the Ordinate at Right Angles, mH is the Radius.

2. In Curves related to a fixed Point, let the Curve $BM=z$, Ordinate $BM=y$, and $Mr=\dot{x}$. Put the Equation of the Curve into Fluxions, and make some of the first Fluxions invariable, for which write 1; and the Equation being turned into Fluxions again, write 1 for the invariable Fluxion, as before: Thus you will determine the other Fluxion and second Fluxion, whose Values being substituted into one of the Forms below which contains that first and second Fluxion, you will have the Radius of Curvature.

FORMS.

$$\text{If } \dot{x}=1, \text{ Radius } MC = \frac{y \times \sqrt{1+\dot{y}^2}^{\frac{3}{2}}}{1+\dot{y}^2-y\ddot{y}} = \frac{y\dot{z}^2\sqrt{\dot{z}^2-1}}{\dot{z}\sqrt{\dot{z}^2-1}-y\ddot{z}}$$

$$\text{If } \dot{y}=1, \text{ Radius } MC = \frac{y \times \sqrt{1+\dot{x}^2}^{\frac{3}{2}}}{\dot{x}+\dot{x}^3+y\ddot{x}} = \frac{y\dot{z}^2\sqrt{\dot{z}^2-1}}{\dot{z}^3-\dot{z}+y\ddot{z}}$$

$$\text{If } \dot{z}=1, \text{ Radius } MC = \frac{y\sqrt{1-\dot{x}^2}}{\dot{x}\sqrt{1-\dot{x}^2}+y\ddot{x}} = \frac{y\sqrt{1-\dot{y}^2}}{1-\dot{y}^2-y\ddot{y}}$$

$$\text{Also } ME = \frac{MC \times \dot{x}}{\dot{z}}, \text{ and } EC = \frac{MC \times \dot{y}}{\dot{z}}.$$

But

FIG. But when you know the Perpendicular upon the Tangent drawn to any Point of the Curve, put this Perpendicular $= u$; and find the Value of $\frac{yy}{u}$, from the Equation of the Curve: And this is the Radius of Curvature.

EXAMPLE I.

49. Let the Equation of the Curve be $ax - ay = x^2 + y^2$ where $AB = x$, $BM = y$, $a =$ a given Line. Let $\dot{x} = 1$ then putting the Equation into Fluxions $a - ay' = 2x + 2yy'$, and again $-a\ddot{y} = 2 + 2y'' + 2yy''$; whence $y' = \frac{a-2x}{a+2y}$, and $\ddot{y} = \frac{-2-2y''}{a+2y} = \frac{-2 \times a + 2y' - 2 \times a - 2y'}{a+2y} = \frac{-4a}{a+2y}$. Whence the Radius of Curvature $= \frac{1 + y'^2}{-\ddot{y}} = \frac{1 + \left(\frac{a-2x}{a+2y}\right)^2}{\frac{4a}{a+2y}} = \frac{(a+2y)^2 + (a-2x)^2}{4a(a+2y)}$. $\frac{2a^2}{4aa} = \frac{a}{\sqrt{2}}$ a determinate Quantity: Therefore

Am is the Arch of a Circle whose Radius is $\frac{a}{\sqrt{2}}$

Ex. 2.

50. Let Am be an Ellipsis, $a =$ Transverse, $b =$ Parameter, and $abx - bxx = ayy$. In Fluxions $ab - 2bx = 2ayy'$ (where $\dot{x} = 1$), and this again in Fluxions $= 2ay'' + 2ayy''$; hence $y' = \frac{ab - 2bx}{2ay}$, and $-\ddot{y} = \frac{b + ay''}{ay} = \frac{4abyy' + ab - 2bx^2}{4aay^3} = (\text{expunging } x) \frac{bb}{4ay^3}$. Therefore $mH = \frac{1 + y'^2}{-\ddot{y}} = \frac{4aayy' + ab - 2bx^2}{-\ddot{y} \times 4aay^3}$

$\frac{abb + 4aayy - 4abyy}{aabb}y = y + \frac{4a - 4b}{abb}y^3$, and Ra- F I G.

50.

$$\text{us } mC = \frac{1 + \dot{y}^2}{-\ddot{y}} = \frac{abb + 4aayy - 4abyy}{2bba^3}.$$

And if Am were an Hyperbola, it would be found in the same Manner that $mH = y + \frac{4a + 4b}{abb}y^3$, and $mC =$

$$\frac{abb + 4a^2y^2 + 4aby^2}{2bba^3}.$$

Or thus in particular Numbers.

Let $a = 3$, $b = 1$, and assume $x = 1$, $\dot{x} = 1$, then

$$= \sqrt{bx - \frac{b}{a}xx} = \sqrt{\frac{2}{3}}, \text{ and by the foregoing operation } \dot{y} = \frac{ab - 2bx}{2ay} = \frac{1}{6\sqrt{\frac{2}{3}}}, \text{ and } -\ddot{y} = \frac{b + a\dot{y}^2}{ay}$$

$$= \frac{3\sqrt{3}}{8\sqrt{2}}. \text{ Whence } mC = \frac{1 + \dot{y}^2}{-\ddot{y}} = \frac{125}{54}.$$

Otherwise thus:

Let Transverse $= 2r$, Conjugate $= 2c$, B the Focus, $BM = y$, $BT = u$ the Perpendicular on the Tan-

57.

gent at M . By the Conic Sections $u = \frac{cy}{\sqrt{2ry - yy}}$,

$$\text{and } \dot{u} = \frac{cry\dot{y}}{2ry - yy^{\frac{3}{2}}} \text{ whence } -\frac{y\ddot{y}}{\dot{u}} = \frac{(2ry - yy)^{\frac{3}{2}}}{cr}, \text{ the}$$

Radius of Curvature in M .

Ex. 3.

In the Parabola $ax = yy$, then $a\dot{x} = 2y$, and $a\ddot{x} = 2$

52.

(putting $\dot{y} = 1$); whence Radius $mC = \frac{1 + \dot{x}^2}{\ddot{x}}$

$$= \frac{1 + \frac{4y}{aa}}{\frac{2}{a}} = \frac{aa + 4yy}{2aa} = \frac{4 \times mD}{aa}.$$

F f

Other-

Otherwise thus:

FIG. Let $a =$ Latus rectum, B the Focus, $BM = y$

58. $BT = u$, the Perpendicular on the Tangent at M . By

the Nature of the Figure $uu = \frac{1}{2}ay$, and $\dot{u} = \frac{ay}{4\sqrt{ay}}$

whence $\frac{yy}{u} = \frac{4yy\sqrt{ay}}{ay} = \frac{4y\sqrt{ay}}{a} = \frac{8uy}{a}$, the

Radius of Curvature in M .

Ex. 4.

51. Let Dm be an Hyperbola between the Asymptotes

$AB = x$, $Bm = y$. Draw mR parallel to the other

Asymptote; then since the Position of the Ordinate

mR is given, there is given the Ratio of mR and RB

to mB perpendicular to AB : Let $RB = sy$, $mR = ty$

$AR = x - sy$. By the Nature of the Figure $AR \times RB$

$= aa = txy - tsyy$, which turned into Fluxions $tx +$

$ty\dot{x} - 2tsy = 0$, and again $2tx + ty\ddot{x} - 2ts = 0$; hence

$\dot{x} = \frac{2sy - x}{y}$, and $\ddot{x} = \frac{2s - 2\dot{x}}{y} = \frac{2x - sy}{yy}$

Then $\frac{1 + \dot{x}^2}{\ddot{x}} = \frac{yy + 2sy - x^2}{2xy - 2syy}$, the Radius of

Curvature in m .

And in a right angled Hyperbola, $s = 0$, and the

Radius becomes $\frac{yy + xx}{2xy} = \frac{yy + xx}{2aa} = \frac{Am^3}{2aa}$.

Ex. 5.

52. Let Am be the Cissoid, whose Equation is $axx - yx^2$

$= y^3$, then $a - y = \frac{y^3}{xx}$, in Fluxions $-1 = \frac{3yy}{xx} - \frac{2y^2\dot{x}}{x^3}$,

reduced $-x^3 = 3y^2x - 2y^3\dot{x}$, in Fluxions $-3x^2\dot{x} = 6yx +$

$3y^2\dot{x} - 6y^2\dot{x} - 2y^3\ddot{x}$; hence $\dot{x} = \frac{3xy^2 + x^3}{2y^3}$, and $\ddot{x} =$

$$\frac{6xy + 3xx - 3yy \times \dot{x}}{2y^3} = \frac{3xy^4 + 6x^3y^2 + 3x^5}{4y^6}. \text{ Then FIG. 52.}$$

$$\text{The Radius} = \frac{1 + \dot{x}^2}{\ddot{x}} = \frac{4y^6 + 9x^2y^4 + 6x^4y^2 + x^6}{2y^3 \times 3xy^4 + 6x^3y^2 + 3x^5}$$

$$= (\text{expunging } y^3, y^6) \frac{4aa - 8ay + 10yy + xx + 9ay - 9yy}{6aay} \times x$$

$$= \frac{4aa + ay + y^2 + x^2}{6aay} \times x = (\text{expunging } xx)$$

$$\frac{4a - 3y}{6 \times a - y} \times a\sqrt{y}. \text{ Therefore in the Vertex A, the}$$

radius of Curvature is o.

Or thus definitively in Numbers.

Let $a=10$, $y=1$, and assume $y=2$, then $x=1$, and $\dot{x}$

$$= \frac{3xy^2 + x^3}{2y^3} = \frac{13}{16}. \text{ And } \ddot{x} = \frac{6xy + 3x^2 - 3yy \times \dot{x}}{2y^3}$$

$$= \frac{75}{16 \times 16}; \text{ whence } \frac{1 + \dot{x}^2}{\ddot{x}} = \frac{85}{48} \sqrt{17}.$$

Ex. 6.

Let $x=y^n$ be an Equation for all Parabolas, then if $y=1$ $\dot{x} = ny^{n-1}$, $\ddot{x} = nn - n \times y^{n-2}$. Therefore

$$\frac{1 + \dot{x}^2}{\ddot{x}} = \frac{1 + nny^{2n-2}}{n.n - 1.y^{n-2}}.$$

Hence it will easily appear that every Parabola, except the Appolonian, has the Radius of Curvature in the Vertex either infinite or nothing at all.

COR. If $ax=yy$, $bx=y^3$, $cx=y^4$, $dx=y^5$, &c. be Series of Parabolas, then the Angle of Contact (made with the Tangent and Curve) at the Vertex of any one is infinitely greater than the Angle of Contact of the next following one. The Angle of Contact of the first $ax=y^2$, is of the same kind with that of Circles.

F f 2

Now

F I G. Now it appears that the Angles of Contact of one kind infinitely exceed those of another kind, since the Radius of Curvature of one kind is infinitely greater than that of another; and the Curvature or Angle of Contact is reciprocally as that Radius. Thus in the

Curve $cx=y^4$, the Radius $= \frac{c}{12yy} = \text{Infinity}$, and

in the Curve $dx=y^5$, the Radius $= \frac{d}{20y^3} = \text{Infinity}$

ty, because $y=0$; and the latter Radius is infinitely greater than the former, for $\frac{d}{20y^3} = \frac{c}{12yy} \times \frac{12d}{20y^3}$

and therefore the Angle of Contact of the first infinitely exceeds the Angle of Contact of the last. Therefore a Curve of one kind, how great soever it may be, cannot be interposed at the Point of Contact between the Curve of another kind and its Tangent however small that Curve may be: Or an Angle of Contact of one kind cannot necessarily contain an Angle of Contact of another kind, as a whole contains a part.

Likewise in this Series of Parabolic Curves $ax=y^{\frac{1}{2}}$, $bx=y^{\frac{2}{3}}$, $cx=y^{\frac{3}{4}}$, $dx=y^{\frac{4}{5}}$, &c. the Angle of Contact which any Curve makes with the Abscissa at the Vertex is infinitely greater than the next preceding. The Angle of Contact of the first $ax=y^{\frac{1}{2}}$ of the same kind with Circles. And though the Angles of Contact of the succeeding Curves do infinitely exceed the preceding ones, yet they can never arrive at the Magnitude of right lined Angles. Moreover between the Angles of Contact of any two of them may other Angles of Contact be found, *ad infinitum* that will infinitely exceed each other.

Ex. 7.

54. Let Dm be the Logarithmic Curve, whose Equation is $yx=ay$, putting $AB=x$, $Bm=y$, $a = \text{Subtangent}$
The

then (if $\dot{x}=1$) $\dot{y} = \frac{y}{a}$, and $\ddot{y} = \frac{\dot{y}}{a} = \frac{y}{aa}$; FIG.

54.

therefore $\frac{1 + \dot{y}^2}{-\ddot{y}} = \frac{aa + yy}{-ay}$, the Radius of Curvature; where the negative Sign only shews its position.

Ex. 8.

Let *Am* be the Catenary, $AB=x$, $Bm=y$, $Am=z$; Equation is $zz=2ax+xx$, then (putting $\dot{z}=1$) Fluxion is $z=ax+xx$, and again $1=a\ddot{x}+x\ddot{x}+$

52.

; hence $\dot{x} = \frac{z}{a+x}$, and $\ddot{x} = \frac{1 - \dot{x}^2}{a+x} =$

$\frac{1 - \frac{z^2}{(a+x)^2}}{a+x} = \frac{aa - z^2}{(a+x)^3}$. Therefore $\frac{\sqrt{1 - \dot{x}^2}}{\ddot{x}} =$

$\frac{(a+x)^3 \sqrt{aa - z^2}}{aa - z^2} = \frac{(a+x)^3}{aa} \sqrt{aa} = \frac{(a+x)^3}{a}$

Ex. 9.

Let *Am* be the Cycloid, $AB=x$, $Bm=y$, Arch AD

55.

$=v$, $AR=a$, then $BD=\sqrt{ax-xx}$, and $\dot{v} =$

$\frac{a}{2\sqrt{ax-xx}}$, by the Nature of the Circle (putting $\dot{x}=1$). By the Property of the Cycloid, $y=v +$

$\sqrt{ax-xx}$, whence $\dot{y} = \dot{v} + \frac{a-2x}{2\sqrt{ax-xx}} = \frac{a-x}{\sqrt{ax-xx}}$

$= \sqrt{\frac{a-x}{x}}$; and $\ddot{y} = \frac{-a}{2x^{\frac{3}{2}}\sqrt{a-x}}$; whence

the Radius $= \frac{1 + \dot{y}^2}{-\ddot{y}} = 2\sqrt{aa-ax} = 2DR$.

Ex. 10.

EX. 10.

FIG. Let Dm be the Conchoid, $RA=b$, $AD=c$, $AB=$

53. $Bm=y$; its Equation is $b+y\sqrt{cc-yy}=xy$. The

squared and divided by yy , $\frac{bbcc}{yy} + \frac{2bcc}{y} + cc =$

$bb - 2by - yy = xx$. In Fluxions, $-\frac{2bbcc}{y^3}$

$\frac{2bcc}{yy} - 2b - 2y = 2x\dot{x}$; and again $\frac{6bbcc}{y^4}$

$\frac{4bcc}{y^3} - 2 = 2\dot{x}^2 + 2x\ddot{x}$; whence $\dot{x} = \frac{-bbcc}{y^3x}$

$\frac{bcc}{yyx} - \frac{b}{x} - \frac{y}{x}$, and $\ddot{x} = \frac{3bbcc}{y^4x} + \frac{2bcc}{y^3x} - \frac{1}{x}$

$\frac{\dot{x}^2}{x}$. Now to find the Radius of Curvature in any

given Point m ; let $b=5$, $c=10$, if you take $y=6$

then $x=14\frac{2}{3}$, and $\dot{x}=-2\frac{3}{72}$, and $\ddot{x}=\frac{499}{1856}$; whence

$\frac{\sqrt{1+\dot{x}^2}^{\frac{3}{2}}}{\ddot{x}} = 40,326$ the Radius of Curvature in the

Point m .

The Radius of Curvature may also be expressed

indefinitely in the same Manner as in the foregoing

tho' more prolixly. Let $\dot{x} = \frac{p}{x}$, $\ddot{x} = \frac{q}{x} - \frac{\dot{x}^2}{x}$

$\frac{qxx - pp}{x^3}$, then $\frac{\sqrt{1+\dot{x}^2}^{\frac{3}{2}}}{\ddot{x}} = \frac{\sqrt{xx+pp}^{\frac{3}{2}}}{x^3}$. Whence

$\frac{\sqrt{1+\dot{x}^2}^{\frac{3}{2}}}{\ddot{x}} = \frac{\sqrt{xx+pp}^{\frac{3}{2}}}{qxx - pp}$; therefore either x or

being given, the other will be known, and thence

the Quantities p, q . For Example, in the Vertex

where $x=0$, the Radius becomes $\frac{p^3}{-pp} = -p =$

$$\frac{bcc}{y^3} + \frac{bcc}{yy} + b + y = \frac{bb}{c} + 2b + c = \text{FIG.}$$

Ex. II.

Let *Dm* be the Quadratrix, $AB=x$, $Bm=y$, AF or $R=r$, $AE=u$, $CE=v$, Arch $CF=t$, $RCF=q$, $C=q-t$, $Am=s$. By the Nature of the Circle,

Fluxion of RC ($-i$) is to the Fluxion of AE as r to v , that is, $-vi = r\dot{u}$, likewise $r\dot{v} =$ and by the Property of the Figure $t=y$, and $i=\dot{y}$; and by similar Triangles $uy = vx$, in Fluxions $y\dot{u} + u\dot{y} = x\dot{v} + v\dot{x}$, and expunging $\dot{u}$, v , we get $-y^2\dot{y} + rx\dot{y} - x^2\dot{y} = ry\dot{x} = -y^2 +$ $-xx$ (putting $\dot{y} = 1$), this in Fluxions $ry\ddot{x} + r\dot{x}$ $- 2y + r\dot{x} - 2x\dot{x}$. Therefore $\dot{x} = \frac{-y^2 + rx - xx}{ry}$

$$\frac{rx - ss}{ry}; \text{ and } \ddot{x} = \frac{-2y - 2x\dot{x}}{ry} = \frac{2ssx - 2ssr}{rryy};$$

$$\frac{\sqrt{1 + \dot{x}^2}^{\frac{3}{2}}}{\ddot{x}} = \frac{\sqrt{rryy + rx - ss}^{\frac{3}{2}} \times rryy}{r^3y^3 \times 2ssx - 2ssr} = \frac{-s}{2ry}$$

$$\frac{\sqrt{rr - 2rx + ss}^{\frac{3}{2}}}{r - x}.$$

If the Radius of Curvature is required in the vertex F , it will be found by expressing AE and CE in infinite Series in Terms of y , rejecting the superfluous Terms. Thus it will hereafter be shewn

(§. 5. Prob. VIII.) that $CF = CE + \frac{CE^3}{6rr} \&c.$

It is $y = v + \frac{v^3}{6rr}$, whence by Reversion of Series

FIG. Series $v = y - \frac{y^3}{6rr}$, and thence $u = \sqrt{rr - vv}$
56.

$r - \frac{yy}{2r}$, whence $uy = vx$ becomes $ry - \frac{y^3}{2r}$

$yx - \frac{y^3}{6rr}$, and $x = r - \frac{yy}{3r}$, and $\dot{x} = -\frac{2y\dot{y}}{3r}$

But the Radius of Curvature in F is $\frac{yy}{\dot{x}}$, where

you must write $r = x$ for x , $-\dot{x}$ for $\dot{x}$. Then

$$\frac{yy}{\dot{x}} = \frac{yy}{2y\dot{y}} \times 3r = \frac{3}{2}r.$$

Ex. 12.

59. Let AM be the Logarithmic Spiral; TM a Tangent; BT , BP perpendicular to BM TM . If $BM = y$, $BP = u$, the given Ratio of MT

BT as c to t ; then $u = \frac{ty}{c}$, and $\dot{u} = \frac{t\dot{y}}{c}$

whence $MC = \frac{yy}{\dot{u}} = \frac{cy}{t}$; therefore C falls on the Line TB produced.

Ex. 13.

60. Let BM be Archimedes' Spiral, $AB = a$, $AD = v$, $BM = y$, and let $by = av$, whence $\dot{b}y$

$a\dot{v}$; but by similar Sectors $\dot{v} = \frac{ax}{y}$ or $\dot{v} = \frac{ax}{y}$

therefore $aax = by\dot{y}$. Let $y = 1$, then $\dot{x} = \frac{by}{aa}$

$\ddot{x} = \frac{b\dot{y}}{aa}$; whence the Radius = $\frac{y \times 1 + \dot{x}^2}{x + x^2 + y^2}$

$$\frac{a^4 + bbyy^{\frac{3}{2}}}{b^2y^2 + 2ba^4} = \frac{a \times a^2 + vv^{\frac{3}{2}}}{bv^2 + 2ba^2}.$$

Ex. 14.

 Suppose the Equation of the Spiral to be $by^m = a^mv$, 60.

 Then, if $\dot{x} = 1$, $mby^{m-1}\dot{y} = a^m\dot{v} = \frac{a^{m+1}\dot{x}}{y}$, or $mby^m\dot{y}$
 $= a^{m+1}$; again, in Fluxions, $m^2by^{m-1}\dot{y}^2 + mby^m\ddot{y} = 0$;

 Hence $\dot{y} = \frac{a^{m+1}}{mby^m}$, and $\ddot{y} = \frac{-a^{2m+2}}{mb^2y^{2m+1}}$. Whence

$$\text{The Radius} = \frac{y \times 1 + \dot{y}^2}{1 + \dot{y}^2 - y\ddot{y}} = \frac{m^2b^2y^{2m} + a^{2m+2}}{m^2b^2y^{2m-1} + m+1 \times mby^{m-1}a^{2m+2}}.$$

Ex. 15.

 Let BM be the reciprocal or hyperbolic Spiral, Radius 61.

 BM or $BD = a$, given Arch $AE = n$, Arch $AF = v$,

 $BM = y$. By the Property of the Curve $an = vy$,

 Hence $y\dot{v} + v\dot{y} = 0$, and $\dot{v} = \frac{-v\dot{y}}{y} = \frac{-a\dot{y}}{y}$:

 But, by similar Triangles, $\dot{v} = \frac{a\dot{x}}{y}$, therefore $y\dot{x} =$
 $-ny$, let $\dot{x} = 1$, then $\dot{y} = \frac{-y}{n}$, and $\ddot{y} = \frac{-\dot{y}}{n} =$
 $\frac{y}{n^2}$. Therefore $\frac{y \times 1 + \dot{y}^2}{1 + \dot{y}^2 - y\ddot{y}} = \frac{y \times n^2 + y^{\frac{3}{2}}}{n^3}$

 (because the Subtangent $BT = n$) $\frac{y \times TM^3}{n^3} =$
 $\frac{y \times GM^3}{y^3} = \frac{GM^3}{y^2}$, supposing BG perpendicular to BM .

G g

Ex. 16.

Ex. 16.

FIG. 62. Let BM be a kind of Spiral, Radius $BD=a$, Arc $DF=t$, $BM=y$, Arch $MR=v$, and let its Nature $a^2y=v^3$. By the similar Sectors BMR and BFD ,
 $v = \frac{ty}{a}$, whence $a^2 = ty^2$, in Fluxions $3y^2\dot{y} = 2t\dot{y}y$, by similar fluxionary Triangles $i = \frac{a\dot{x}}{y}$

and expunging i , $3a\dot{x} + 2t\dot{y} = 0$, let $\dot{y}=1$, then $\dot{x} = \frac{-2t}{3a}$, and $\ddot{x} = \frac{-2\dot{t}}{3a} = \frac{-2\ddot{x}}{3y} = \frac{4t}{9ay}$

$$\frac{4v}{9yy} : \text{Therefore } \frac{y \times \sqrt{1 + \dot{x}^2}^{\frac{3}{2}}}{\dot{x} + \dot{x}^3 + y\ddot{x}} = \frac{y \times 9yy + 4v^2}{-v \times 6yy + 8a^2}$$

$$= \frac{-v^3}{a^4} \times \frac{9v^4 + 4a^4}{6v^4 + 8a^4}.$$

Or thus, in particular Numbers.

Let $y=1$, $v=4$, $\dot{y}=1$, then $\dot{x} = \frac{-2t}{3a}$

$$\frac{-2v}{3y} = \frac{-8}{3}, \text{ and } \ddot{x} = \frac{-2\dot{x}}{3y} = \frac{16}{9}. \text{ When}$$

$$\frac{y \times \sqrt{1 + \dot{x}^2}^3}{\dot{x} + \dot{x}^3 + y\ddot{x}} = \frac{-73\sqrt{73}}{536} \text{ the Radius of Curvature, where the negative Sign relates only to the Position of it.}$$

SCHOLIUM.

63. Let there be any geometrical Curve defined by the general Equation $e + fx^m + gy^n + bx^py^q = 0$; make $BD = x$, $BM = y$, the Perpendicular to the Curve $MQ = z$. Then the Radius of Curvature may be expressed by algebraic Terms affected with π , thus; compute the Value

Value of $\ddot{y}$ in the foregoing general Equation (put- F I G.
 $\dot{x}$ invariable), which substitute in the Quantity 63.

$\frac{\pi}{y}$, and put $\frac{\pi}{y}$ for $\frac{\dot{z}}{\dot{x}}$; and the Radius of Cur-
 vature will be =

$$\frac{(ngy^{n-1} + sbx'y^{s-1})^3}{\dots} \times \pi^3$$

$$\left. \begin{aligned} & m.n-1.fx^{m-2} + r.r-1.bx^{r-2}y^s \times ngy^{n-1} + sbx'y^{s-1} \\ & 2rsbx^{r-1}y^{s-1} \times mfx^{m-1} + rbx^{r-1}y^s \times ngy^{n-1} + sbx'y^{s-1} \\ & n.n-1.gy^{n-2} + s.s-1.bx'y^{s-2} \times mfx^{m-1} + rbx^{r-1}y^s \end{aligned} \right\} \times y^3$$

EXAMPLE.

Let $bx^2 - ay^2 + abx = 0$, represent an Hyperbola,
 Transverse, b = Parameter. Comparing this with
 the general Form, we shall find $e=0, f=b, m=2, g=$
 $-a, n=2, h=ab, r=1, s=0$, whence the Radius

$$\text{Curvature} = \frac{-2ay^3 \times \pi^3}{2b \times (-2ay)^2 - 2a \times 2bx + ab^2} \times y^3$$

$$= (\text{expunging } x) \frac{(-2ay)^3 \pi^3}{-2a^2b^2y^3} = \frac{4\pi^3}{bb}. \text{ And}$$

will be the same for the Ellipsis.

P R O B. VI.

To determine the (Quality or Degree of) Variation
 of Curvature in any Point of a Curve.

The Variation of Curvature depends on the Mo-
 ment of the Radius of Curvature; and the Moment

F I G. of the Curve; and it is as the Moment of the Radius of Curvature if the Moment of the Curve is given; and reciprocally as the Moment of the Curve if the Moment of the Radius is given. Therefore this Variation is as the Fluxion of the Radius of Curvature divided by the Fluxion of the Curve: Therefore,

1. Find the Radius of Curvature by the last Problem which divide by the Fluxion of the Curve; and the Fluxions may be exterminated by the Equation of the given Curve, or perhaps by expressing their Ratio by the Help of the Tangent, Ordinate, or Subnormal.

2. In Curves referred to an Axis, finding the Fluxion of the Quantity $\frac{\dot{z}^3}{-\ddot{y}}$, which divide by $\dot{z}$, putting $\dot{x}=1$, and exterminating $\dot{z}$, $\ddot{z}$, you will get the following Rule. Put x for the Abscissa, y for the Ordinate, $\dot{x}=1$: Then substitute the Values of $\dot{y}$, $\ddot{y}$, $\ddot{\ddot{y}}$ (gathered from the Equation of the Curve) into the Quantity
$$\frac{-3\dot{y}\ddot{y}^2 + \ddot{\ddot{y}} \times 1 + \dot{y}^2}{\ddot{y}^2}$$
, and it will give the Variation.

EXAMPLE 1.

52. Let the Curve be a Parabola, then $ax=yy$, and $\dot{y} = \frac{a}{2y} \ddot{y} = \frac{-a\dot{y}}{2yy}$, $\ddot{y} = \frac{2a\dot{y}^2 - a\dot{y}\ddot{y}}{2y^3}$. Now if $a=2$, $x=\frac{1}{2}$, $y=1$, then $\dot{y}=1$, $\ddot{y}=-1$, $\ddot{\ddot{y}}=3$, whence
$$\frac{-3\dot{y}\ddot{y}^2 + \ddot{\ddot{y}} \times 1 + \dot{y}^2}{\ddot{y}^2} = 3.$$
 Likewise if $x=2$, $y=2$ then the Variation = 6.

In general,

$$\begin{aligned} \text{Since } ax = y^2, \text{ therefore } \dot{y} &= \frac{a}{2y}, \text{ and } \ddot{y} = \frac{-a\dot{y}}{2yy} \\ &= \frac{-aa}{4y^3}, \text{ and } \ddot{\ddot{y}} = \frac{2a\dot{y}^2 - a\dot{y}\ddot{y}}{2y^3} = \frac{3a\dot{y}}{4y^4} = \frac{3a^2}{8y^5} \end{aligned}$$

Whence

Et. II. of FLUXIONS.

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FIG.

52.

hence
$$\frac{-3\dot{y}\ddot{y}^2 + \dot{y} \times \overline{1 + \dot{y}^2}}{\ddot{y}^2} = \frac{-3a^5}{32y^7} + \frac{3a^3}{8y^5}$$

$$\frac{3a^5}{32y^7} : \times \frac{16y^6}{a^4} = \frac{6y}{a}.$$

Ex. 2.

In the Ellipsis let $bx - \frac{b}{a}xx = yy$. Suppose $a = 50$.

$b=2$, to find the Variation of Curvature when $x = \frac{1}{2}$, and $y = \frac{1}{2}$. By the Process of Example 2. of

last Problem, $\dot{y} = \frac{ab-2bx}{2ay} = -1$, $\ddot{y} =$

$\frac{-ay^2}{ay} = -8$, and it will be found that $\dot{y} =$

$\frac{-ay^2 - 2ay\ddot{y}}{ay} \times y = -48$. Whence $\frac{-3\dot{y}\ddot{y}^2 + \dot{y} \times \overline{1 + \dot{y}^2}}{\ddot{y}^2}$

the Variation required.

Or in general thus:

By the last Problem the Radius of Curvature is,

$\frac{(b+4aay-4aby)^{\frac{3}{2}}}{2bba^3}$, and its Fluxion divided by

$\frac{a-b}{bba\dot{z}} \times 6y\dot{y}\sqrt{aabb+4aay-4aby}$. But from

Equation of the Curve $\frac{\dot{y}}{\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}} =$

$\frac{ab-2bx}{aabb+4aay-4aby}$; whence the Variation of

Curvature becomes $\frac{a-b}{baa} \times \frac{1}{2}a-x \times 12y$; and is

before every where as the Rectangle mBO , sup-

ing O the Center.

Ex. 3.

Ex. 3.

FIG. Let $x = y^3$ denote a cubic Parabola; then $3y\dot{y} =$
 52. and $\dot{y} = \frac{1}{3y^2}$, $\ddot{y} = \frac{-2\dot{y}}{3y^3} = \frac{-2}{9y^5}$, $\dot{\ddot{y}} = \frac{10\dot{y}}{9y^6} =$
 $\frac{10}{27y^8}$; whence $-3\dot{y} + \frac{\ddot{y}}{\dot{y}^2} \times \overline{1+\dot{y}^2} = \frac{15y^2}{2} -$
 the Variation required.

Ex. 4.

55. In the Cycloid Am the Radius of Curvature
 $2\sqrt{aa-ax}$, and its Fluxion $= \frac{-a\dot{x}}{\sqrt{aa-ax}}$; but
 the Property of the Figure $\dot{z} = \frac{AD \times \dot{x}}{x} = \frac{DR}{y}$
 $= \frac{\dot{x}\sqrt{aa-ax}}{y}$, whence $\frac{-a\dot{x}}{\dot{z}\sqrt{aa-ax}} = \frac{-ay}{aa-ax}$
 $\frac{-y}{BR} = -\frac{AB}{BD}$. The negative Sign only shews
 that the Curvature increaseth.

Ex. 5.

59. In the Logarithmic Spiral, (by Ex. 12. Prob. 1.)
 the Radius of Curvature is $\frac{cy}{t}$ (putting $c:t:t:d$
 $TM:TB:BM$), and its Fluxion is $\frac{c\dot{y}}{t}$, and $\dot{z}$
 $\frac{c\dot{y}}{d}$; therefore $\frac{c\dot{y}}{t\dot{z}} = \frac{d}{t}$ for the Variation of Cur-
 vature, which therefore is uniform.

Ex. 6.

61. Let BM be the hyperbolic Spiral, (Ex. 15. Prob. 1.)
 the Radius of Curvature is $\frac{y \times \overline{n^2 + y^2}^{\frac{3}{2}}}{n^3}$, where
 Fluxion

Fluxion is $\frac{n^2 + 4y^2}{n^3} \dot{y} \sqrt{n^2 + y^2}$, but the Subtangent FIG. 61.

$=n$, therefore $\dot{z} = \frac{\dot{y}}{y} \sqrt{nn + yy}$, by which di-

the other Quantity, then $\frac{n^2 y + 4y^3}{n^3} =$ Vari-

on of Curvature in M .

P R O B. VII.

*find the Nature of the Curve, by whose Evolu-
tion a given Curve is described.*

A Curve ADM is said to be described by the Evo-
lution of another Curve VEC , when a right Line is
applied to be apply'd to the convex Side of the
Curve VEC , and the End of it A is made to move
with a circular Motion, and to describe the Curve
 ADM ; whilst the Part DE or MC that continually
touches the Curve VEC is extended into a right Line.
The Curve VEC is called the Evolute of ADM .

Hence it appears that the Line CM which is a
Tangent at C to the Curve VEC , is the Radius of
Curvature in M of the Curve ADM , and is equal in
Length to the Curve CE + the right Line ED the
Tangent of E , or to the Length of the Curve
 EV + the Line VA . Now the Evolute VEC being
the Locus of the Center of Curvature of all the
Points of the Curve ADM , the Nature of the Curve
 VEC is to be found. Therefore

*In Curves related to an Axis, put the Abscissa
 $B=x$, perpendicular Ordinate $BM=y$, and let CM
be the Radius of Curvature, and draw CH parallel to
 AB ,*

64.

F I G. *AB, and CL to BM, and produce MB to H, and v, u for the Abscissa and Ordinate of the Curve VC ferred to the Axis AL: Then find MH, HC by Prob. and expunging x and y, the Nature of the Curve VC be discovered.*

47. 2. *And in Spirals find the Radius of Curvature by Prob. V. which will give all the Points C in Evolute.*

Thus we shall have $EC = \frac{MC \times \dot{y}}{\dot{z}}$, $ME = \frac{MC}{\dot{z}}$

and therefore $BE = y - \frac{MC \times \dot{x}}{\dot{z}}$, and then BC

$$\sqrt{yy + MC^2 - \frac{2y\dot{x}}{\dot{z}} \times MC}.$$

COROL. I. Hence it will be easy to find any Number of Curves whose Lengths may be expressed by finite Equations; by assuming any Curve *AM* at Pleasure and finding its Evolute *EC*: For the Radius of Curvature *MC* will give the Length of Evolute; or $MC - DE = \text{Length of the Part EC of the Curve.}$

64. **COR. 2.** Hence also if the Curve *VEC* be given, Curve *AM* described by its Evolution may be found.

For let $AV = b$, $VEC = s$, then $AV + VC = CM$

$+s$, and by similar Triangles $\dot{s} : \dot{u} :: b + s : MH$
 $\frac{b+s}{\dot{s}} \dot{u}$, therefore MB or $y = \frac{b+s}{\dot{s}} \dot{u} - u$. All

$: \dot{v} :: b + s : \frac{b+s}{\dot{s}} \dot{v} = HC$ or BL , therefore AB

$= b + v - \frac{b+s}{\dot{s}} \dot{v}$ where $\dot{s} = \sqrt{\dot{v}^2 + \dot{u}^2}$. If the

evolution begins in *V*, $b = 0$. Now if the Value $\dot{v}$ or $\dot{u}$ be got from the Equation of the Curve, and substituted in these Equations, you'll have the Nature of Curve *AM*, without any functional Quantities. *expung*

panging v , u when it can be done, you'll have the F I G.
 equation between x and y . But as s is always found
 herein, 'tis a Sign that the Nature of the Curve AM
 depends on the Rectification of the Curve VE ; whose
 value, when it can be had, may be substituted in its
 room.

EXAMPLE I.

Let AM be a Parabola, $ax=yy$, and, putting $\dot{y}=1$, 64.

we shall find (by Prob. V.) $CH = \frac{1+\dot{x}^2}{\ddot{x}} = \frac{1}{2}a +$

x , and $MH = \dot{x} \times \frac{1+\dot{x}^2}{\ddot{x}} = CH \times \dot{x} = y + \frac{4xy}{a}$

$y + \frac{4y^3}{aa}$, and when x and y is 0, $AV = \frac{1}{2}a$.

Let $v=VL$, $u=LC$, and we have $v = (AB - AV +$
 $L = x - \frac{1}{2}a + \frac{1}{2}a + 2x = 3x$, and $u = (MH - MB =)$

$\frac{4y^3}{aa} = \frac{4x\sqrt{ax}}{a}$: Whence $x^3 = \frac{v^3}{27} = \frac{auu}{16}$; or

$\frac{6v^3}{27a} = uu$, for the Property of the Curve VC , which

therefore is a semicubical Parabola whose Latus rec-
 tum is $\frac{27}{16}a$.

Ex. 2.

Let GM be the Logarithmic Curve, $AG=1$, $AB=x$, 69.

$M=y$, Subtangent $RB=a$, then $\dot{a}y=y\dot{x}$, and (by
 Prob. V.) $MH = \frac{-aa-yy}{y}$, and $HC = \frac{-aa-yy}{a}$.

Let $v=AD=x - \frac{aa+yy}{a}$, $u=DC=2y + \frac{aa}{y}$, whence

$\dot{x} = \frac{-2y\dot{y}}{a} = \frac{aa-2yy}{ay}\dot{y}$; and $\dot{u} = 2\dot{y} - \frac{aa\dot{y}}{yy} =$

$\frac{2yy-aa}{yy}\dot{y}$; therefore $\dot{y} = \frac{ay\dot{u}}{aa-2yy} = \frac{-yy\dot{u}}{aa-2yy}$, or $\dot{a}v$

H h =

F I G. $= -y\ddot{u}$, the Equation of the Curve wherein the Center C is always found.

Now since $\dot{v} = \frac{aa - 2yy}{ay} \dot{y}$, therefore v will increase when aa is greater than $2yy$, and decrease when $2yy$ exceeds aa . Likewise u will increase when $2yy$ exceeds aa , that is when v decreases; and it will decrease when v increases.

Ex. 3.

65. Let AM be the Cycloid $AB = x$, $BM = y$, Arc $AD = s$, Axis $TR = c$, $AR = a$, then $y = s + \sqrt{a - x}$, $\dot{y} = \sqrt{\frac{a-x}{x}}$, $\ddot{y} = \frac{-a}{2x\sqrt{ax-xx}}$; whence $MH = \frac{1+\dot{y}^2}{-\ddot{y}} = 2\sqrt{ax-xx}$, $CH = \dot{y} \times MH = 2a - 2y$. Let $RN = v$, $NC = u$; then $v = a - x$, and $u = y - 2\sqrt{ax-xx} = s - \sqrt{ax-xx}$, that is (expunging s) $u = s - \sqrt{av - vv}$: And when $x = 0$, $RP = a$. Hence the Curve TCP is also a Cycloid equal to the Cycloid AMT . For completing the Parellelogram RTQ and describing the Semicircle TSQ , $NP = (a - v) = AB$, $ES = (\sqrt{av - vv} =) BD$, Arch $AD = Q$ and $EC = (c - u = c - s + \sqrt{av - vv} = EN - Q + ES =) TS + ES$: Therefore TC is the Cycloid.

Ex. 4.

64. Suppose AM to be the Catenary, $AB = x$, $BM = z$, and $z = \sqrt{2ax + xx}$, and $\dot{z} = \frac{z}{a+x}$, $\ddot{z} = \frac{aa}{a+x^3}$ (putting $\dot{z} = 1$) whence $MH = \frac{\dot{z}\sqrt{1-\dot{z}^2}}{\ddot{z}} = \frac{z \times a + x}{a}$; and $CH = \frac{1-\dot{z}^2}{\ddot{z}} = a + x$, and when $x = 0$

Let $AV = a$. Let $VL = v$, $LC = u$, and we have $F I G.$

$$= 2x, \text{ and } u = \frac{z \times a + x}{a} - y = \frac{a+x}{a} \sqrt{2ax+xx} \quad 64.$$

$$y, \text{ whence } u = \frac{2a+v}{2a} \sqrt{av + \frac{1}{4}vv} = 2.30258a$$

$$\text{Logarithm of } \frac{2a+v+\sqrt{av+\frac{1}{4}vv}}{2a}, \text{ because } y =$$

$2.30258 \times$ into that Log.

Ex. 5.

Suppose VM to be a Curve whose Tangent MT is = 66.

$$\text{the given Line } a, AB = x, BM = y, VM = z, \dot{y} = 1, \text{ then}$$

$$\text{the Nature of the Curve } \dot{y} = \frac{-y\ddot{x}}{\sqrt{aa-yy}} = \frac{-y\ddot{z}}{a};$$

$$\text{hence } \ddot{z} = \frac{-a}{y}, \ddot{z} = \frac{a}{yy}: \text{ Then } MH = \frac{aa-yy}{-y},$$

$HC = -\sqrt{aa-yy}$, the negative Signs show they must

be taken upward (and to the Right Hand): But in

the Vertex V , MH and HC are nothing, therefore the

Evolution begins in V .

$$\text{Let } VL = v, LC = u, \text{ then } v = \frac{aa-yy}{y} + y - a =$$

$$\frac{aa-ay}{y}, \text{ and } y = \frac{aa}{a+v}: \text{ Also } u = x + \sqrt{aa-yy}, \text{ and}$$

$$= \dot{x} - \frac{y\dot{y}}{\sqrt{aa-yy}} = \frac{-\dot{y}}{y} \sqrt{aa-yy} - \frac{y\dot{y}}{\sqrt{aa-yy}}$$

$$= \frac{-a\dot{y}}{y\sqrt{aa-yy}} = (\text{expunging } y \text{ and } \dot{y}) \frac{a\dot{v}}{\sqrt{2av+vv}}, \text{ an}$$

Equation to the Catenary, therefore VC is the Cate-

nary Curve.

Ex. 6.

Let AM be the Cissoid whose Equation is $ax^2 - yx^2 =$ 70

10 , and suppose $a = 10$. To find the Point C of the

H h 2

Evolute

F I G. Evolute when $y=2$, $x=1$. By a former Calcula

70. (Ex. 5. Prob. V.) $\dot{x} = \frac{3xy + x^3}{2y^3} = \frac{13}{16}$, and

$$\frac{6xy + 3xx - 3yy \times \dot{x}}{2y^3} = \frac{75}{16 \times 16}; \text{ whence (by Prob. V.) } HC = \frac{1 + \dot{x}^2}{\ddot{x}} = 5\frac{2}{3} = BL, \text{ and } MH = \dot{x} \times$$

$= 4\frac{2}{3}$, and $LC = 2\frac{2}{3}$. And so for any other given Point of the Curve.

Ex. 7.

67. Let BM be the Log. Spiral, TBC perpendicular to BM ; and let $c:t:s::TM:TB:BM$. Then

Ex. 12. Prob. V.) $MC = \frac{cy}{t}$, and $BC = \frac{sy}{t}$. The

fore since the Angle BCM is equal BMT , the Curve BC is also a Log. Spiral the same with BM .

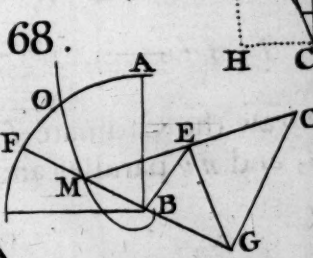
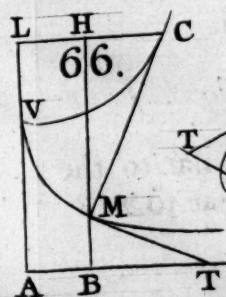
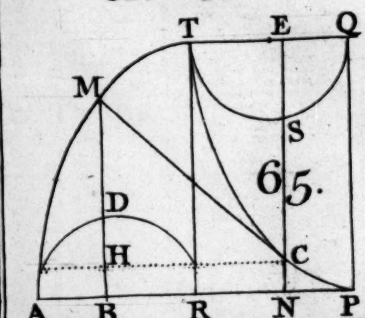
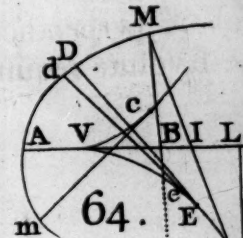
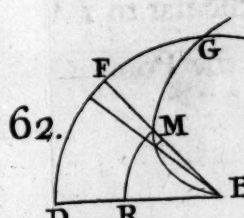
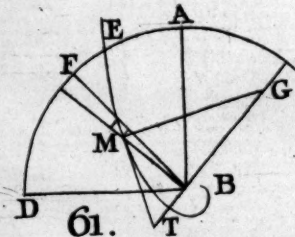
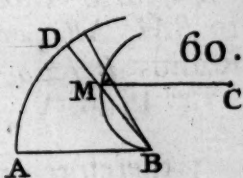
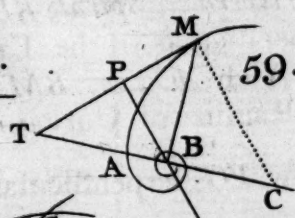
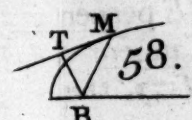
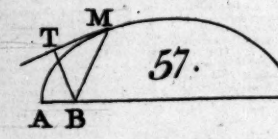
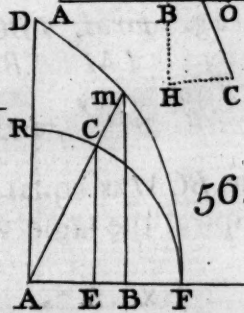
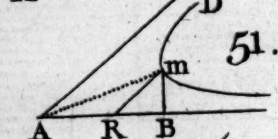
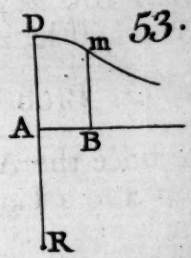
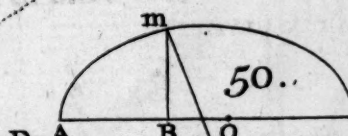
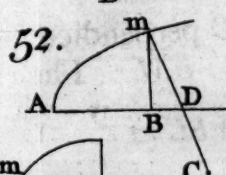
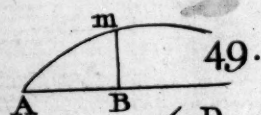
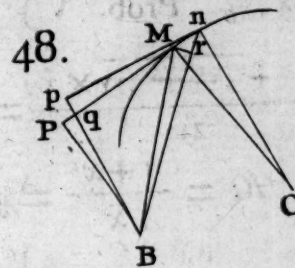
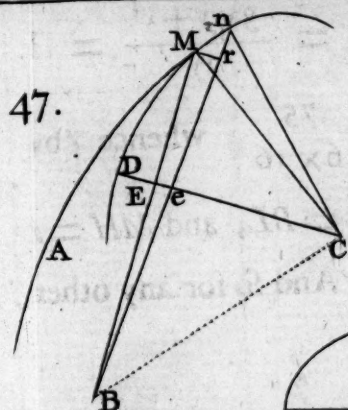
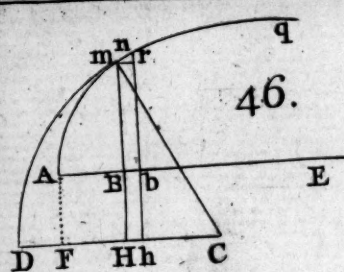
Ex. 8.

68. Suppose BM be an Hyperbolic Spiral, BE perpendicular to BM , MC perpendicular to the Tangent in M ; let Radius $BA=a$, Arch $AO=n$, $BM=y$, then (Ex. 15. Prob. V.) Radius of Curvature in M $\frac{EM}{yy}$; therefore draw EG perpendicular to EM and GC perpendicular to GM , and the Point C is the Evolute required.

P R O B. VIII.

To determine the Length of Curve Lines.

71. Draw the Ordinate BM perpendicular to the Axis AB , and nb parallel and infinitely near to MB . And



Spirals draw Bn infinitely near BM ; and let Mr be perpendicular to nB or nb . Let the Abscissa $B = x$, Ordinate $BM = y$, Curve $AM = z$, $Mr =$

I. G. 72.

$rn = \dot{y}$, $Mn = \dot{z}$. In the right angled Triangle Mrn , $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$, therefore $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; wherefore

By the Nature of the Curve exterminate $\dot{x}$ or $\dot{y}$ out of the Equation $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; and the Fluent will give z the Length of the Curve.

EXAMPLE 1.

Let the Nature of the Curve be $\frac{2}{3aa} \times \overline{aa + xx}^{\frac{3}{2}} = y$. 73.

Then its Fluxion is $\dot{y} = \frac{2x\dot{x}}{aa} \sqrt{aa + xx}$: Then $\dot{z} =$

$$\sqrt{\dot{x}^2 + \dot{y}^2} = \dot{x} \sqrt{1 + \frac{4xx}{aa} + \frac{4x^4}{a^4}} = \dot{x} + \frac{2x^2\dot{x}}{aa}$$

By Form 1st.) the Fluent $z = x + \frac{2x^3}{3aa}$, which needs Correction.

Ex. 2.

In the common Parabola $ax = yy$. Here $\dot{x} = \frac{2y\dot{y}}{a}$, 74.

Hence $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}}{a} \sqrt{aa + yy}$: And by

Form the 9th and 13th, the corrected Fluent $z = \frac{y}{a}$

$$+ \frac{2.30258a}{4} \times \text{Log.} \frac{y + \sqrt{\frac{1}{4}aa + yy}}{\frac{1}{2}a}$$

Ex. 3.

In the semicubical Parabola $ax^2 = y^3$, or $x = \frac{y^{\frac{2}{3}}}{a^{\frac{1}{2}}}$, 74.
and

F I G. and $\dot{x} = \frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$: And therefore $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \dot{y}$
74.

$\sqrt{1 + \frac{9y}{4a}}$, whose Fluent, by Form the 3d, is $z = \frac{8a}{27}$

$\times 1 + \frac{9y}{4a}$. But in the Vertex, y and $z = 0$; there-

fore (Prob. 12. Sect. I.) the Fluent corrected is $z =$

$$\frac{8a}{27} \times 1 + \frac{9y}{4a}^{\frac{3}{2}} - \frac{8a}{27}.$$

Ex. 4.

74. Let $ax^m = y^{m+1}$ be an Equation for various Parabolas.

Then $x = \frac{y^{\frac{m+1}{m}}}{a^{\frac{1}{m}}}$, and $\dot{x} = \frac{m+1}{ma^{\frac{1}{m}}} y^{\frac{1}{m}} \dot{y}$, and $\dot{x}^2 =$

$$\left(\frac{m+1}{ma^{\frac{1}{m}}} y^{\frac{1}{m}} \dot{y} \right)^2 = b y^{\frac{2}{m}} \dot{y}^2 \quad \left(\text{putting } b = \frac{m+1}{ma^{\frac{1}{m}}} \right).$$

Whence $\dot{z} = \dot{y} \sqrt{1 + b y^{\frac{2}{m}}} = y^{\frac{1}{m}} \dot{y} \sqrt{b + y^{-\frac{2}{m}}}$.

Now the Fluent of $y^{\frac{1}{m}} \dot{y} \sqrt{b + y^{-\frac{2}{m}}}$ will be had by Form the 15th in finite Terms, when m is any positive even Number.

And likewise the Fluent of $\dot{y} \sqrt{1 + b y^{\frac{2}{m}}}$ will be found by Form the 11th, when m is any odd Number; finding first the Fluent of $y^{\frac{1}{m} - 1} \dot{y} \sqrt{1 + b y^{\frac{2}{m}}}$ by Form 9th and 13th.

Ex. 5.

Ex. 5,

To find the Length of the Arch of the Circle *AM*. FIG.

Let Radius *AC* = *r*, Sine *BM* = *y*, *AB* = *x*, by the Nature of the Circle $2rx - xx = yy$, and $\dot{x} =$ 75.

$$\frac{y\dot{y}}{r-x} = \frac{y\dot{y}}{\sqrt{rr-yy}}, \text{ therefore } \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} =$$

$$\frac{r\dot{y}}{\sqrt{rr-yy}}; \text{ and, by Form the 16th, } z = y +$$

$$\frac{y^2}{2.3.rr} A + \frac{3.3yy}{4.5rr} B + \frac{5.5.yy}{6.7.rr} C, \text{ \&c. where } A,$$

B, *C*, &c. are the whole foregoing Terms.

Otherwise thus,

Let *AC* = *r*, Tangent *AT* = *t*, *Tt* = *i*, *AM* = *z*, 76.

Mn = *z*, then *CT* = $\sqrt{rr+tt}$. By the similar

Triangles *CAT* and *Ttr*, $Tr = \frac{rt}{\sqrt{rr+tt}}$; and by

the similar Triangles, *CMn*, *CTr*, $\dot{z} = \frac{rr\dot{t}}{rr+tt}$, or

$$\dot{z} = \frac{rr\dot{t}}{rr+tt} = \dot{t} - \frac{t^2\dot{t}}{rr} + \frac{t^4\dot{t}}{r^4} - \frac{t^6\dot{t}}{r^6}, \text{ \&c.}$$

whence (by Form the 16th) the Fluent $z = t -$

$$\frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8}, \text{ \&c.}$$

Now if *r* = 1, and Arch *AM* = 30°, then *t* = $\sqrt{\frac{1}{3}}$:

$$\text{And then } z = \sqrt{\frac{1}{3}} \times 1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3}$$

$$+ \frac{1}{9.3^4} - \text{\&c.} = \frac{1}{12} \text{ the Circumference : And half}$$

$$\text{the Circumference} = \frac{\sqrt{12}}{1} - \frac{\frac{1}{3}A}{3} - \frac{\frac{1}{3}B}{5} - \frac{\frac{1}{3}C}{7}$$

FIG. — $\frac{\frac{1}{3}D}{9}$ &c. where $A, B, C, \&c.$ are the preceding Numerators with their Signs.

Or thus, by collecting every two Terms into one, we shall have half the Circumference = $\frac{1.6}{3}\sqrt{3} \times$ into

$\frac{1}{1.3} + \frac{2}{5.7.9} + \frac{3}{9.11.9^2} + \frac{4}{13.15.9^3}, \&c.$ that is putting $\frac{1.6}{3}\sqrt{3} = A, \frac{1}{3}A = B, \frac{1}{3}B = C, \frac{1}{3}C = D, \&c.$

then half the Circumference = $\frac{1}{1.3}A + \frac{2}{5.7}B +$

$\frac{3}{9.11}C + \frac{4}{13.15}D + \frac{5}{17.19}E + \frac{6}{21.23}F + \&c.$

which Series will be as follows.

3,079201	435678	004077	38
58651	455917	676268	14
3455	893867	203147	11
259	930478	900749	53
21	794996	295857	48
1	943349	566200	68
	180259	667580	92
	17186	683703	87
	1672	167602	79
	165	238193	86
	16	530067	76
	1	670177	49
		170147	80
		17454	05
		1801	06
		186	80
		19	46
		2	04
			22
			2

3.141592 653589 793238 46 &c.
= Half the Circumference.

Ex. 6.

Let FMD be the Quadrant of an Ellipsis, $AD=a$, F I G. $AF=c$, $AB=x$, $BM=y$, $FM=z$. Then by the 77.

Nature of the Curve $y = \frac{c}{a} \sqrt{aa-xx}$, and $\dot{y} =$

$$\frac{-cx\dot{x}}{a\sqrt{aa-xx}}; \text{ whence } \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}\sqrt{a^4 - aaxx + c^2xx}}{a\sqrt{aa-xx}}$$

$$= (\text{putting } d = \frac{aa-cc}{aa}) \frac{\dot{x}\sqrt{aa-dxx}}{\sqrt{aa-xx}} = \frac{\dot{x}\sqrt{a^2-dxx} \times aa-xx}{aa-xx}$$

$$= \dot{x} + \frac{1-d}{2aa} x^2 \dot{x} + \frac{3-2d-dd}{8a^4} x^4 \dot{x} \text{ \&c.}; \text{ whence}$$

$$z = x + \frac{1-d}{6aa} x^3 + \frac{3-2d-dd}{40a^4} x^5 + \frac{5-3d-dd-d^3}{112a^6} x^7$$

+ \&c. for the Arch FM.

$$\text{But for the Quadrant FD, } \dot{z} = \frac{\dot{x}\sqrt{aa-dxx}}{\sqrt{aa-xx}} =$$

$$\frac{\dot{x}}{\sqrt{aa-xx}} \times a - \frac{dxx}{2a} - \frac{ddx^3}{2.4a^3} - \frac{3d^3x^6}{2.4.6a^5} \text{ \&c. but}$$

the Fluent of $\frac{\dot{x}}{\sqrt{aa-xx}} =$ Arch of the circumscrib-

ing Quadrant of the Circle divided by $a = \frac{\mathcal{Q}}{a}$;

whence the Quadrant FD or $z = \mathcal{Q} \times 1 - \frac{d}{2.2} +$

$$\frac{3d}{4.4} A + \frac{3.5d}{6.6} B + \frac{5.7d}{8.8} C + \text{\&c.}, \text{ by Form 17th.}$$

Ex. 7.

To find the Length of the Arch of the Hyperbola. 78.
Let $a =$ half the Transverse, $c = AF$ half the conju-
gate,

F I G. gate, $AB = x$, $BM = y$: Then $y = \sqrt{cc + \frac{cc}{aa}xx}$,
78.

$$\text{and } \dot{y} = \frac{cx\dot{x}}{\sqrt{a^4 + aaxx}}, \text{ and } \dot{x}^2 + \dot{y}^2 = \frac{aa + \frac{aa+cc}{aa}xx}{aa+xx} \dot{x}^2;$$

$$\text{put } d = \frac{aa+cc}{aa}, \text{ and then } \dot{z} = \dot{x} \sqrt{\frac{aa+dx}{aa+xx}} =$$

$$\dot{x} \sqrt{1 - \frac{1-d}{aa}x^2 - \frac{1-d}{a^4}x^4 - \frac{1-d}{a^6}x^6} = \dot{x} \times 1 - \frac{1-d}{2aa}x^2 + \frac{3-2d-dd}{8a^4}x^4 - \frac{5-3d-dd-d^3}{16a^6}x^6 +$$

$$\text{\textcircled{E}c; therefore } z = x - \frac{1-d}{6aa}x^3 + \frac{3-2d-dd}{40a^4}x^5 - \frac{5-3d-dd-d^3}{112a^6}x^7 \text{\textcircled{E}c for the Arch FM.}$$

Ex. 8.

79. Let AM be the Cycloid, $AB = x$, $BM = y$, $AR = a$,
 $BD = v$, Arch $AD = s$, then $y = s + v$ By the Nature
of the Circle $vv = ax - xx$, and $\dot{v} = \frac{a\dot{x} - 2x\dot{x}}{2v}$, and

$$\dot{s} = \frac{a\dot{x}}{2v}; \text{ therefore } \dot{y} = \dot{s} + \dot{v} = \frac{a-x}{v}\dot{x}, \text{ whence}$$

$$\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \dot{x} \sqrt{1 + \frac{a-x}{vv}} = \frac{\dot{x}}{v} \sqrt{aa - ax}$$

$$= \frac{\dot{x} \sqrt{aa - ax}}{\sqrt{ax - xx}} = \dot{x} \sqrt{\frac{a}{x}}. \text{ Therefore } z = 2\sqrt{ax} \\ = 2 \text{ Cord } AD.$$

Ex. 9.

69. Let GM be the Log. Curve, $AB = x$, $BM = y$, $AG = b$,
Subtangent $BR = a$, then $a\dot{y} = y\dot{x}$, therefore $\dot{z} =$

$$\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}}{y} \sqrt{aa + yy} = \frac{y\dot{y}}{\sqrt{aa + yy}} + aay^{-2}\dot{y}$$

$\frac{aay^{-2}\dot{y}}{\sqrt{1+aay^{-2}}}$. Whence (by Form the 3d and 9th) F I G. 69.

$$z = \sqrt{aa + yy} - 2.30258a \times \text{Log.} \frac{a + \sqrt{aa + yy}}{y};$$

and corrected (by Prop. XII.) z or $GM = \sqrt{aa + yy}$
 $-\sqrt{aa + bb} + 2.302585a \times \text{Log.} \frac{ay + y\sqrt{aa + bb}}{ab + b\sqrt{aa + yy}}.$

Ex. 10.

In the Cissoid DM , let $AD=a$, $AB=x$, $BM=y$, 80.

then $y = \frac{a-x^2}{\sqrt{ax-xx}} = \frac{(a-x)^{\frac{3}{2}}}{x^{\frac{1}{2}}}$, and $\dot{y} =$

$\frac{-a\dot{x} - 2x\dot{x}}{2xx} \sqrt{ax-xx}$, whence $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \frac{a\dot{x}}{2x} \sqrt{\frac{a+3x}{x}} = \frac{ax^{-\frac{3}{2}}\dot{x}}{2} \sqrt{a+3x}$, and

(by Form the 9th, 13th and 12th) $z = 2.3025a\sqrt{3}$
 $\times \text{Log.} \sqrt{3x + \sqrt{a+3x}} : -a\sqrt{\frac{a+3x}{x}}$. And

the Fluent corrected, z or $MD = a\sqrt{\frac{a+3x}{x}} - 2a$

$+ 2.302585a\sqrt{3} \times \text{Log.} \frac{\sqrt{3a + \sqrt{4a}}}{\sqrt{3x + \sqrt{a+3x}}}.$

Ex. 11.

Let AM be the Quadratrix, $AB=x$, $BM=y$, Arch 81.
 $AK=t$, Sine $KG=v$, $AC=a$. Then (by Ex. 5th)

$t = v + \frac{v^3}{6aa} + \frac{3v^5}{40a^4} + \frac{5v^7}{112a^6} + \text{Ec.} = y$, by

the Nature of the Quadratrix. Then by Reversion

of Series $v = y - \frac{y^3}{6aa} + \frac{y^5}{120a^4} - \frac{y^7}{5040a^6} \text{Ec.}$

I i 2

then

FIG. then $\sqrt{aa-vv} = a - \frac{yy}{2a} + \frac{y^4}{24a^3} - \frac{y^6}{720a^5} \&c.$
 81. $= CG$; and by similar Triangles $(KG) v : GC ::$
 $(MB) y : (BC) a - x = a - \frac{y^2}{3a} - \frac{y^4}{45a^3} - \frac{2y^6}{945a^5}$
 $\&c.$ whence $x = \frac{yy}{3a} + \frac{y^4}{45a^3} + \frac{2y^6}{945a^5} \&c.$ and
 $\dot{x} = \frac{2y\dot{y}}{3a} + \frac{4y^3\dot{y}}{45a^3} + \frac{4y^5\dot{y}}{315a^5} \&c.$ therefore $\dot{z} =$
 $\sqrt{\dot{x}^2 + \dot{y}^2} = \dot{y} \times 1 + \frac{2y^2}{9aa} + \frac{14y^4}{405a^4} + \frac{604y^6}{127575a^6}$
 $\&c.$ and $z = y + \frac{2y^3}{27a^2} + \frac{14y^5}{2025a^4} + \frac{604y^7}{893025a^6}$
 $\&c.$

Ex. 12.

82. Let BM be Archimedes's Spiral, whose Equation is
 $rv = cy$, putting $v = \text{Arch } ED$. Then $r\dot{v} = c\dot{y}$, but
 (by similar Triangles) $\dot{v} = \frac{r\dot{x}}{y}$, whence $\dot{x} = \frac{cy\dot{y}}{rr}$:
 Therefore $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}}{rr} \times \sqrt{r^4 + ccyy}$; and
 (by Form 9th and 13th) the Fluent (corrected) is $z =$
 $\frac{y}{2rr} \sqrt{r^4 + ccyy} + \frac{2.302585rr}{2c} \times \text{Log.} \frac{cy + \sqrt{r^4 + ccyy}}{rr}$.

Ex. 13.

82. 59. Let BM be the Log. Spiral, and let c, t, s express
 the Ratio of the Tangent TM , Subtangent TB , and
 Ordinate BM . Then $\dot{z} = \frac{c\dot{y}}{s}$ and the Fluent $z =$
 $\frac{cy}{s} = TM$. Therefore the Length of the Spiral
 MAB making an infinite Number of Revolutions is
 equal to the Tangent MT .

Ex. 14.

Ex. 14.

Let BM be the hyperbolic Spiral, $AB=a$, Arch AO FIG. 83.
 $=b$, Arch $AD=v$, $MB=y$, then $ab=vy$, and thence

$$vy + y\dot{v} = 0, \text{ and } y\dot{v} = -v\dot{y} = \frac{-aby}{y}; \text{ and by}$$

$$\text{similar Triangles } -y\dot{v} = a\dot{x}, \text{ and thence } \dot{x} = \frac{b\dot{y}}{y};$$

$$\text{therefore } \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}}{y} \sqrt{bb + yy} =$$

$$\frac{bby^{-2}\dot{y}}{\sqrt{1 + bby^{-2}}} + \frac{yy\dot{y}}{\sqrt{bb + yy}}. \text{ And the Fluent (by}$$

$$\text{Form the 3d and 9th) is } z = \sqrt{bb + yy} - b \times$$

$$2.302585 \text{ Log. } \frac{b + \sqrt{bb + yy}}{y}: \text{ Whence taking}$$

$$\text{any Ordinate } BQ = d, \text{ the Length of the Arch}$$

$$QM = \sqrt{bb + dd} - \sqrt{bb + yy} - 2.30258b \times \text{Log.}$$

$$\frac{by + y\sqrt{bb + dd}}{bd + d\sqrt{bb + yy}}.$$

$$\frac{by + y\sqrt{bb + dd}}{bd + d\sqrt{bb + yy}}.$$

P R O B. IX.

To transform Spirals into geometrical Curves, or
 geometrical Curves into Spirals of equal Length.

Let ADd be a geometrical Curve related to the
 Axis AB ; let the perpendicular Ordinates BD, bd ,
 be infinitely near together; and let the Points B, b
 be supposed to approach one another, and to coincide
 in B , whilst the Particle of the Curve Dd remains
 the same; then the Parallelogram $BDdb$ will be
 changed into the Triangle BDd . In like Manner let
 all

84.

F I G. all the Points in AB be supposed to be contracted into one Point B , and then the Figure ADd will be transform'd into the Spiral QDd , whose Center is B , in which the corresponding Ordinates are equal, and the Triangles Ddr in both Figures similar and equal; and the Area of the Figure ABD twice the Area of the Spiral BQD .

85. Again let ADd be a Spiral, ABD its Complement, BD , bd , two Arches of Circles infinitely near each other, whose Center is A ; AH a geometrical Curve (to the Abscissa $AB=AD$) equal in Length to the Spiral: Let BH , bb be drawn infinitely near each other, and HK parallel to AB . Then the Triangles DdC and HbK are similar and equal, therefore $dC=Kb$. Now let AB or $AD=y$, $BH=u$, Arch $BD=v$, and by similar Triangles $y:v::Ab:bC::Bb:bC-v$
 $=\frac{v}{y}\times Bb$. Therefore $bC+\frac{v}{y}\times Bb-v$ or $bd-v=Kb$
 $+\frac{v}{y}\times Bb$: Therefore (because the Moments are as the Fluxions) $\dot{v}=u+\frac{vy}{y}$.

84. 1. Therefore in Curves referr'd to an Axis, let $AB=x$, $BD=y$, Radius BR or $BT=b$, Arch $RT=v$, $Dr=x$, $rd=y$, $TS=v$. In the Curve AD , expunge x and $\dot{x}$ out of the Equation of the Curve, by Help of the Equation $b\dot{x}=y\dot{v}$; and the Fluent will give the Nature of the Spiral. Likewise in the Spiral QD , expunge all Quantities except x and y and their Fluxions out of the Equation $b\dot{x}=y\dot{v}$, by Help of the Equation of the Curve; and the Fluent will show the Nature of the geometrical Curve AD .

85. 2. In the Spiral AD , where there is given the Relation of AD (y) to the Arch BD (v): By help of the Equation of the Spiral, exterminate v and $\dot{v}$ out of the Equation

Equation $\dot{v} = \dot{u} + \frac{v\dot{y}}{y}$, and the Fluent gives the Nature of the Curve *AH*. Or by the Equation of the Curve *AH*, expunge *u*, and *u* out of the Equation $\dot{v} = \dot{u} + \frac{v\dot{y}}{y}$, and the Fluent gives the Nature of the Spiral. FIG. 85.

EXAMPLE I.

Let $ax = yy$ denote a Parabola *ADd*, then we have 84.
 $\dot{x} = \frac{2y\dot{y}}{a} = \frac{y\dot{v}}{b}$, and $2by = a\dot{v}$; whence the
 Fluent $2by = av$; therefore the Spiral *QD* is that of
Archimedes.

Ex. 2.

Let *AD* be a Circle, $2ax - xx = yy$, Arch *AD* = *z*, 86.
 therefore $\dot{x} = \frac{y\dot{y}}{a-x} = \frac{y\dot{y}}{\sqrt{aa-yy}} = \frac{y\dot{v}}{b}$, therefore
 $\dot{v} = \frac{b\dot{y}}{\sqrt{aa-yy}}$, let $b=a$, then $\dot{v} = \frac{a\dot{y}}{\sqrt{aa-yy}} = \dot{z}$
 therefore $v = z$, that is *RT* = *AD*, and *BD* = *BD*
 = *TF*; draw *GD*, then since *BG* = *BT*, and *BD* = *TF*,
 and $\angle DBG = \angle FTB$, therefore $\angle GDB = \angle BFT =$
 a right Angle, and the Curve *BDG* is a Semicircle.

Ex. 3.

Let $aa = xy$ be the Equation of an equilateral Hyper- 87.
 bola between the Assymptotes, then $x\dot{y} + y\dot{x} = 0$, and $\dot{x}$
 $= -\frac{x\dot{y}}{y} = -\frac{aa}{yy} \dot{y} = -\frac{y\dot{v}}{b}$. Therefore $-\frac{baa\dot{y}}{y^3}$
 $= \dot{v}$, and thence $v = \frac{baa}{2yy}$, or putting $b = 2a$,
 $a = vyy$, for the Nature of the Spiral *QDd*.

Ex. 4.

Ex. 4.

FIG. 88. Let ED be the Logarithmic Curve, whose Equation is $y\dot{x} = ay$, then $\dot{x} = \frac{a\dot{y}}{y} = \frac{y\dot{v}}{b}$, and $\frac{a\dot{y}}{yy} = \dot{v}$, therefore $v = -\frac{ab}{y}$: Therefore any Portion of the Arch RT or v is $= ab \times \frac{DB - EA}{DB \times EA}$; BD , EA being the corresponding Ordinates.

Ex. 5.

Let $ay = bx$ denote a plain Triangle, then $a\dot{y} = b\dot{x} = y\dot{v}$, therefore $\frac{a\dot{y}}{y} = \dot{v}$, consequently $v = a \times \text{Log. } y$, an Equation to the Log. Spiral.

Ex. 6.

84. Let QD be the Spiral of Archimeds, whose Equation is $d\dot{y} = av$; then $\frac{d\dot{y}}{a} = \dot{v} = \frac{b\dot{x}}{y}$, or $d\dot{y}y = ab\dot{x}$, and the Fluent $\frac{dyy}{2} = abx$, or $yy = \frac{2ab}{d}x$, an Equation to a Parabola.

85. Or thus: Let $AB = y$, $BD = v$, then $av = yy$, whence $\frac{2y\dot{y}}{a} = \dot{v} = \dot{u} + \frac{v\dot{y}}{y} = \dot{u} + \frac{yy\dot{y}}{a}$, and $\dot{u} = \frac{yy\dot{y}}{a}$, therefore $2au = yy$, an Equation to the common Parabola, as before.

Ex. 7.

84. Let QD be the Log. Spiral, whose Equation is $a\dot{y} = b\dot{x}$, therefore $ay = bx$ a plain Triangle.

Ex. 8.

Ex. 8.

Let QD be the Hyperbolic Spiral, then $ab=vy$, and F I G.
 $vy + y\dot{v} = 0$, or $\dot{v} = \frac{-vy}{y} = \frac{-aby}{yy} = \frac{b\dot{x}}{y}$, or 88.
 $\frac{-a\dot{y}}{y} = \dot{x}$, whence the Fluent $\frac{-x}{a} = \text{Log. } y$, an
 Equation to the Log. Curve.

Ex. 9.

Suppose $y^3 = av^2$ to denote the Spiral ADd , then 85.
 $\dot{v} = \frac{3y^{\frac{1}{2}}\dot{y}}{2\sqrt{a}} = \dot{u} + \frac{vy}{y} = \dot{u} + \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a}}$; therefore $\dot{u} =$
 $\frac{y^{\frac{1}{2}}\dot{y}}{2\sqrt{a}}$, whence $u = \frac{y^{\frac{3}{2}}}{3\sqrt{a}}$, or $9au^2 = y^3$, a semicubi-
 cal Parabola.

Ex. 10.

If $v = y \sqrt{\frac{a+y}{c}}$ be the Equation of the Spiral; 85.
 then $\dot{v} = \frac{2a\dot{y} + 3y\dot{y}}{2\sqrt{ca + cy}} = \dot{u} + \frac{vy}{y} = \dot{u} + \dot{y} \sqrt{\frac{a+y}{c}}$
 $= \dot{u} + \frac{a\dot{y} + y\dot{y}}{\sqrt{ca + cy}}$; whence $\dot{u} = \frac{y\dot{y}}{2\sqrt{ca + cy}}$, there-
 fore $u = \frac{y-2a}{3c} \sqrt{ca + cy}$.

Ex. 11.

Let $BDdE$ be a Semicircle, $BD=y$, $BE=a$, by 89.
 similar Triangles $Dr = \frac{BD \times rd}{ED}$, or $\dot{x} = \frac{y\dot{y}}{\sqrt{aa-yy}}$,
 whence $x = -\sqrt{aa-yy}$; when corrected $x = a -$
 $\sqrt{aa-yy} = BE - ED$, therefore ADL is the Qua-
 drant of a Circle, whose Radius is a .

K k

Ex. 12.

EX. 12.

FIG. Let QDd be a Parabola, B the Focus, $a =$ Latus
 90. Rectum, $BD = y$, DP a Tangent: By the Nature
 of the Figure the Perpendicular $BP = \frac{1}{2}\sqrt{ay}$, and by
 similar Triangles $Dr = \frac{BP \times dr}{PD}$ or $\dot{x} = \frac{ay}{\sqrt{4ay - aa}}$;

therefore $x = \sqrt{ay - \frac{1}{4}aa} =$ Ordinate DI . Hence
 the Curve will be the same Parabola referr'd to the
 Axis AB ; making $AQ = QB$, and AB perpendicular
 to AQ .

PROB. X.

To find the Areas of Curves.

91. In any Curve AD related to the Abscissa AB ,
 produce the Ordinate DB till BE be equal to a given
 Line, and completing the Parallelogram $ABEC$:
 If the Areas $ACEB$ and ADB be conceived to be
 generated by the right Lines BE , BD , moving along
 the Abscissa AB ; then the Fluxions of the Areas
 $ACEB$ and ABD will be as the describing Lines BE ,
 BD drawn into their Velocities of moving, that is
 into the Fluxions of the Abscissas: Now since $BE \times$
 $AB =$ Area $ACEB$, and therefore the Fluxion of the
 Area $ACEB = BE \times$ Fluxion of AB ; consequently
 the Fluxion of the Area $ABD = DB \times$ Fluxion of
 AB .

92. In like Manner in Curves related to a fixt Point B ,
 the fluxionary Triangle BDM is the Moment of the
 Area, and this is $= BM \times \frac{DR}{2}$, or $=$ the Perpen-

dicular

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dicular BT (on the Tangent at M) $\times$ by $\frac{1}{2}DM$; but DR and DM are as their generating Fluxions. Hence FIG.

1. In Curves related to an Axis AB , let $AB=x$, $BD=y$, Area $ABD=z$, then $\dot{z}=y\dot{x}$; therefore by the Equation of the Curve, expunge one of the Quantities y or $\dot{x}$ out of the Equation $\dot{z}=y\dot{x}$, and finding the Fluent it gives the Value of z the Area. Sometimes it will be necessary to find the Area of the Complement ~~ABD~~ , ABD , but the Rule will still be the same if you make ~~AB~~ the Abscissa. 91.

Note, If the Ordinates are not at right Angles to the Abscissa, the Area before found must be diminished in the Ratio of Radius to the Sine of the true Angle.

2. In Curves related to a fixt Point B , let $BD=y$, Perpendicular BT (on the Tangent) $=p$, Curve $AD=v$, $DR=\dot{x}$. Area $BAD=z$: Then by the Nature of the Curve expunge y or $\dot{x}$ out of the Equation $\dot{z}=\frac{y\dot{x}}{2}$; or expunge p or $\dot{v}$ out of the Equation $\dot{z}=\frac{p\dot{v}}{2}$, and the Fluent will give z the Area. 92.

And if the fluxionary Triangle BDM can be computed any other Way, and the heterogeneous Quantities expung'd by the Equation of the Curve; the Fluent will give the Area as before.

EXAMPLE I.

To find the Area of a Triangle. Let the Base $CD=b$, Perpendicular $AQ=p$, $AB=x$, dD (parallel to CD) $=y$: By similar Triangles $y=\frac{bx}{p}$, therefore $\dot{z}=y\dot{x}=\frac{bx\dot{x}}{p}$, therefore the Fluent $z=\frac{bxx}{2p}$, $=\frac{xy}{2}$, and when $x=p$, and $y=b$, the Area $z=$ 93.

F I G. $\frac{bp}{2}$. Or since $\dot{x} = \frac{py}{b}$, therefore $\dot{z} = y\dot{x} = \frac{py\dot{y}}{b}$,
and $z = \frac{pyy}{2b} = \frac{xy}{2}$, as before.

Otherwise thus :

94. Let the Perpendicular $BC=p$, $BD=y$, $de=\dot{y}$; and
by similar Triangles $Dd = \frac{yy'}{\sqrt{yy-pp}}$, therefore $\dot{z} =$

$$\frac{pyy'}{2\sqrt{yy-pp}} = \text{Area } BDd, \text{ and } \dot{z} = \frac{pyy'}{2\sqrt{yy-pp}}, \text{ and}$$

thence $z = \frac{1}{2} p \sqrt{yy-pp}$. Now in the Point C, y is
a Minimum $= p$. Therefore by the Schol. Prop. XII.

first the Part ABC must be found, which is $\frac{p}{2}$
 $\sqrt{AB^2-pp}$, and then the Part BCD , which will be
 $\frac{p}{2} \times \sqrt{BD^2-pp}$. And the whole $ABD = \frac{p}{2}$
 $\sqrt{AB^2-pp} + \frac{p}{2} \sqrt{BD^2-pp}$, or $\frac{1}{2} AD \times CB$.

Ex. 2.

91. Let ABD be any Parabola, where $x = y^m$; then $\dot{x}$
 $= my^{m-1}\dot{y}$, therefore $\dot{z} = y\dot{x} = my^m\dot{y}$; and $z = \frac{m}{m+1}$
 $y^{m+1} = \frac{m}{m+1} yx$, and if $m=2$, $z = \frac{2}{3}xy$.

Or thus: Let $Ab=x$, $bD=y$, $x = y^{\frac{1}{m}}$, then $\dot{x} =$
 $\frac{1}{m} y^{\frac{1}{m}-1} \dot{y}$, therefore $\dot{z} = y\dot{x} = \frac{1}{m} y^{\frac{1}{m}} \dot{y}$, and thence
 $z =$

$z = \frac{1}{m+1} y^{\frac{1}{m}+1} = \frac{1}{m+1} xy$: And if $m=2$, $z =$ FIG.
 by, the Area AbD .

Ex. 3.

Let FD be an Hyperbola, $CA=a$, $AF=b$, $AB=x$, 95.

$$BD = \frac{ab}{a+x} = y, \text{ Then } \dot{z} = y\dot{x} = \frac{ab\dot{x}}{a+x} = b\dot{x} -$$

$$\frac{bx\dot{x}}{a} + \frac{bx^2\dot{x}}{aa} - \frac{bx^3\dot{x}}{a^3} \text{ \&c. and the Area } ABDF$$

$$\text{or } z = bx - \frac{bx^2}{2a} + \frac{bx^3}{3aa} - \frac{bx^4}{4a^3} + \frac{bx^5}{5a^4}, \text{ \&c.}$$

Or (by Form 4th) the Fluent (corrected) is $z =$
 $2.302585ab \times \text{Log. } \frac{a+x}{a}.$

Ex. 4.

Let FD be any kind of Hyperbola, $CB=x$, $BD=y$, 95.

and suppose $y = \frac{1}{x^n}$, then $\dot{z} = y\dot{x} = x^{-n}\dot{x}$; and the

$$\text{Area } z = \frac{1}{1-n} x^{1-n} = \frac{1}{1-n} xy, \text{ for the Area } CBD.$$

If the Angle ABD is oblique, that Area must be diminished in the Ratio of Radius to the Sine of ABD .

If $n=1$ the Space will be infinite, if n be greater than 1 you get the Space BDE .

Ex. 5.

Let AD be a Circle, $AC=a$, $AB=x$, $BD=y =$ 96.

$\sqrt{ax-xx}$, then $\dot{z} = y\dot{x} = \dot{x} \sqrt{ax-xx}$, whose Fluent

$$\text{(by Form 16th) is } z = \sqrt{ax} \times \frac{2}{3}x - \frac{x^2}{5a} - \frac{x^3}{4.7a^2}$$

$$- \frac{1.3x^4}{4.6.9a^3} - \frac{3.5.x^5}{4.6.8.11a^4} \text{ \&c. } = \frac{2x\sqrt{ax}}{3} - \frac{\frac{x}{2a}A}{5} +$$

FIG. 96. $+\frac{x}{4a}B + \frac{3x}{6a}C + \frac{5x}{8a}D \&c.$ where $A, B, C,$
 are the preceding Numerators. Or (by Form
 11, 13) $z = \frac{aa}{8} \phi + \frac{2x-a}{4} \sqrt{ax-xx}$ (putting
 $= .01745 \times$ twice the Degrees in the Arch whose S
 is $\sqrt{\frac{x}{a}}$, and Radius 1.)

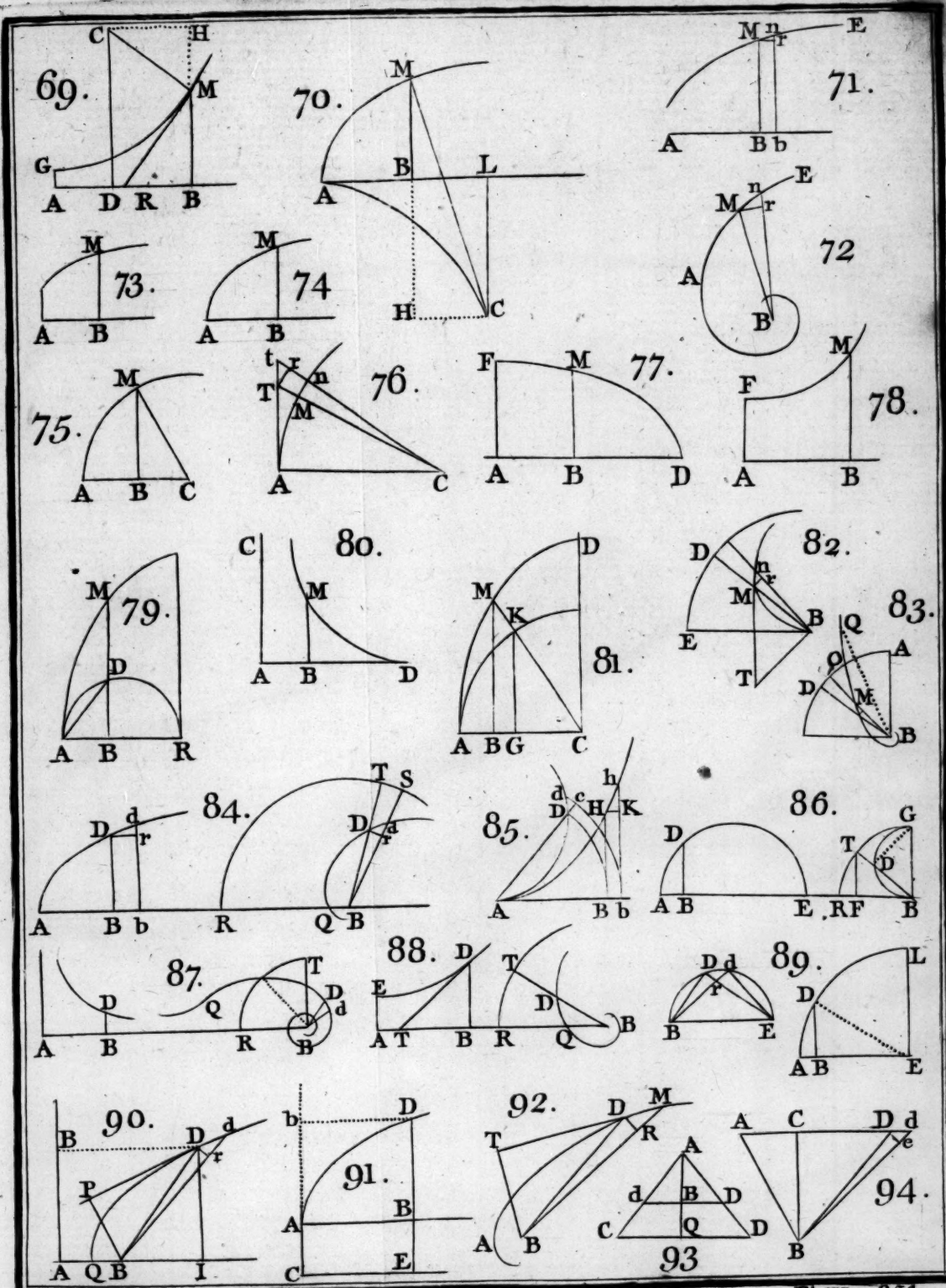
If AD be 60° , then $x = \frac{1}{4}a$, and Area $ABD =$
 $-\frac{3}{2.4.5}A + \frac{1.5}{4.4.7}B + \frac{3.7}{6.4.9}C + \frac{5.9}{8.4.11}D$
 to which adding the Triangle $DBE = \frac{aa}{32} \sqrt{3}$, y
 have $\frac{1}{2}$ the Area of the Circle. Here $A, B, C, \&c.$
 are the preceding Terms.

If $y = \sqrt{ax+xx}$ be an Equation to a right-angle
 Hyperbola, we shall in like Manner find $z = \sqrt{ax+xx}$
 $x: \frac{2}{3}x + \frac{x^2}{5a} - \frac{x^3}{4.7aa} + \frac{3.x^4}{4.6.9a^3} - \frac{3.5x^5}{4.6.8.11a^4}$
 $+ \&c.$ or $z = \frac{a+2x}{4} \sqrt{ax+xx} - \frac{aa}{8} \phi$, where
 $\phi = 2.30258 \text{ Log. } \frac{a+2x+2\sqrt{ax+xx}}{a}$.

Likewise the Area of an Ellipsis is found after the
 same Manner as the Circle. Or if you put c for the
 Conjugate, tis no more than multiplying the former
 Area of the Circle by $\frac{c}{a}$. And thus $\frac{c}{a}$ multiplyed
 into the former Area of the Hyperbola, gives the
 Area of the Hyperbola whose Conjugate is c .

Ex. 6.

97. In the Ellipsis FDE , to find the Area adjoining
 the Center A . Let $AE=a$, $AF=c$, $AB=x$, $BD=y$, then $y = \frac{c}{a} \sqrt{aa-xx}$, by the Nature of the Figure



; then $\dot{z} = y\dot{x} = \frac{cx}{a} \sqrt{aa - xx}$, whence (by Form 16) F I G.

97.

$$\frac{cx}{1} - \frac{\frac{xx}{2aa}A}{3} + \frac{\frac{xx}{4aa}B}{5} + \frac{\frac{3xx}{6aa}C}{7} + \frac{\frac{5xx}{8aa}D}{9}$$

for the Area ABDF. Or $z = \frac{ca\phi}{2} +$

$\sqrt{aa - xx}$, putting $\phi = .017453 \times$ Degrees in the

whose Sine is $\frac{x}{a}$ and Radius 1.

After the same Manner in the Hyperbola, when 98.

$$\frac{c}{a} \sqrt{aa + xx}, \text{ we find } z = \frac{cx}{1} + \frac{\frac{xx}{2aa}A}{3} -$$

$$\frac{B}{5} - \frac{\frac{3xx}{6aa}C}{7} - \frac{\frac{5xx}{8aa}D}{9} \text{ \&c. Or } z = \frac{ca\phi}{2}$$

$$\frac{cx}{2a} \sqrt{aa + xx}, \text{ putting } \phi = 2.30258 \text{ Log.}$$

$$+ \frac{\sqrt{aa + xx}}{a}.$$

Ex. 7.

To find the Area ADB of the Ellipsis ADE, generated 99.
by the Line BD revolving round the Focus B.

Let $2a =$ Transverse AE, $2c =$ Conjugate, BD
draw the Tangent DT and BT perpendicular

$$\text{it; then by the conic Sections } BT = \frac{cy}{\sqrt{2ay - yy}},$$

$$\text{hence } DT = \frac{y \sqrt{2ay - yy - cc}}{\sqrt{2ay - yy}}. \text{ And by similar}$$

Triangles DT : TB :: MR : RD, that is

$$2ay - yy - cc : c :: y : \dot{x} = \frac{cy}{\sqrt{2ay - yy - cc}};$$

whence

FIG. whence $\dot{z} = \frac{y\dot{x}}{2} = \frac{cy\dot{y}}{2\sqrt{-cc+2ay-yy}}$: P

99.

$v = a - y$, $p = aa - cc$; and (by Form 27th) $\dot{z} =$

$$\frac{cv\dot{v}}{2\sqrt{p-vv}} = \frac{cav}{2\sqrt{p-vv}}; \text{ whence (by Form}$$

$$3d \text{ and } 10th) z = \frac{-c}{2}\sqrt{p-vv} - \frac{ca}{2} \times ,0174$$

Degrees in the Arch whose Sine is $\sqrt{\frac{vv}{p}}$, but when

$z=0$, $y=a-\sqrt{p}$, therefore correcting the Fluent, and putting ϕ = Number of Degrees in the Arch whose

Co-sine is $\frac{a-y}{\sqrt{aa-cc}}$, and Radius 1, then we have

$$\text{the Area } BAD \text{ or } z = \frac{,0174533ca\phi}{2} - \frac{c}{2}$$

$\sqrt{-cc+2ay-yy}$. And the Area of the Semi-ellipsis $AME = ,01745ca \times 90 = \frac{1}{2}ca \times 3.141592$.

Ex. 8.

100. To find the parabolic Area BAD , generated by the Line BD revolving round the Focus B .

Let a = Latus Rectum, $BD=y$; by the Nature of the Parabola the Perpendicular $BT = \frac{1}{2}\sqrt{ay} = p$, and by similar Triangles $DT : DB :: MR : MD$, or

$$\sqrt{yy - \frac{1}{4}ay} : y :: \dot{y} : \dot{v} = \frac{y\dot{y}}{\sqrt{yy - \frac{1}{4}ay}}; \text{ whence } \dot{z}$$

$$= \frac{p\dot{v}}{2} = \frac{\sqrt{a}}{4} \times \frac{y\dot{y}}{\sqrt{y - \frac{1}{4}a}}; \text{ and (by Form } 3d$$

$$\text{and } 11th) \text{ the Fluent } z = \frac{a+2y}{12}\sqrt{ay - \frac{1}{4}aa}.$$

Ex. 9.

101. Let AD be an Hyperbola, B the Center, $BA=a$, Semi-conjugate $=b$, $BC=x$, Tangent $AT=t$, Ordinate

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dinate $CD = y$; by similar Triangles $ay = tx$, and by the Nature of the Figure $bbxx - bbaa = aayy = ttxx$, and

F I G. 101.

$xx = \frac{bbaa}{bb - tt}$. The fluxionary Triangle BTt is to

the fluxionary Triangle BDD , as BT^2 to BD^2 or aa to

xx , that is $\frac{at}{2} : \dot{z} :: aa : \frac{bbaa}{bb - tt} :: bb - tt : bb$;

whence $\dot{z} = \frac{\frac{1}{2}bbat}{bb - tt}$, and the Fluent $z = \frac{at}{2} + \frac{at^3}{6bb}$

$+ \frac{at^5}{10b^4} + \frac{at^7}{14b^6} \&c.$ Or (by Form 6th) the Fluent

$z = \frac{1}{4}ab \times 2.30258 \text{ Log. } \frac{b+t}{b-t}$ for the hyperbolic

Sector BAD .

Or thus: Since $t = \left(\frac{ay}{x} = \frac{ay}{\sqrt{aa + \frac{aa}{bb}yy}} = \right.$

$\frac{by}{\sqrt{bb + yy}}$, therefore $\dot{t} = \frac{b^{\frac{3}{2}}\dot{y}}{(bb + yy)^{\frac{3}{2}}}$, therefore

$\dot{z} = \left(\frac{\frac{1}{2}abb\dot{t}}{bb - tt} = \right) \frac{ab\dot{y}}{2\sqrt{bb + yy}}$; and the Fluent $z =$

$\sqrt{bb + yy} \times : \frac{ay}{2b} - \frac{2yy}{3bb} A - \frac{4yy}{5bb} B - \frac{6yy}{7bb} C \&c.$

by Form the 15th. Or (by Form the 9th) $z = \frac{1}{2}ab$

$\times 2.30258 \times \text{Log. } \frac{y + \sqrt{bb + yy}}{b} = \text{Sector } BAD.$

Ex. 10.

Let RD be the Logarithmic Curve, $AB = x$, $BD = y$, $AR = b$, Subtangent $= a$, by similar Triangles $ay = yx = \dot{z}$, whence $z = ay$, and (corrected) Area $ABDR = ay - ab$.

102.

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Ex. 11.

EX. 11.

FIG. 103. Let AD be the Cissoïd of Diocles, $AE=a$, $AB=x$,

$BD=y$, by the Nature of the Curve $y = \frac{x^{\frac{3}{2}}}{\sqrt{a-x}}$;

therefore $\dot{z} = y\dot{x} = x^{\frac{3}{2}}\dot{x} \times \overline{a-x}^{-\frac{1}{2}}$; and the Fluent (by Form the 10th and 11th) is $z = 3$ Sectors $CAM - \frac{3}{4}a - \frac{1}{2}x\sqrt{ax-xx}$, that is $z = 3AMB - 2x\sqrt{ax-xx}$.

EX. 12.

104. Let ED be the Conchoïd of Nichomedes, $CA=b$, $AE=a$, $AB=x$, $BD=y$; the Equation of the Curve is

$\overline{b+y}\sqrt{aa-yy} = xy$, in Fluxions $\dot{y}\sqrt{aa-yy} - \overline{b+y} \times y\dot{y} \times \overline{aa-yy}^{-\frac{1}{2}} = x\dot{y} + y\dot{x}$, therefore $\dot{z} = y\dot{x} =$

$\dot{y}\sqrt{aa-yy} - y\dot{y} \times \frac{b+y}{\sqrt{aa-yy}} (-x\dot{y} =) - \frac{b+y}{y} \times$

$\dot{y}\sqrt{aa-yy} = \frac{-y^2\dot{y}}{\sqrt{aa-yy}} - \frac{baa\dot{y}}{y\sqrt{aa-yy}} =$

$\frac{-y^2\dot{y}}{\sqrt{aa-yy}} - \frac{ba^2y^{-2}\dot{y}}{\sqrt{-1+a^2y^{-2}}}$; whose corrected

Fluent (by Form 10 and 11, and Form 9th) is $z =$

$\frac{1}{2}y\sqrt{aa-yy} + 2.30258ab \times \text{Log.} \frac{a + \sqrt{aa-yy}}{y}$

$+ .017453$ Degrees in the Arch whose Co-sine is $\frac{y}{a}$, Radius = 1.

EX. 13.

105. Let AD be the Cycloid, $AB=x$, $BD=y$, $AE=2r$, $FC=u$, Arch $AC=v$; by the Nature of the Circle

$\frac{2ry-yy}{u} = uu$, and $\dot{u} = \frac{ry-y\dot{y}}{u}$; and $\dot{v} = \frac{r\dot{y}}{u}$. And

by

by the Nature of the Cycloid $x = v + u$, and $\dot{x} = \text{FIG.}$

$$\dot{v} + \dot{u} = \frac{r\dot{y}}{u} + \frac{r\dot{y} - y\dot{y}}{u} = \frac{2r\dot{y} - y\dot{y}}{\sqrt{2ry - yy}}; \text{whence } \dot{z} =$$

$$y\dot{x} = \frac{2ry - yy}{\sqrt{2ry - yy}} \dot{y} = \dot{y} \sqrt{2ry - yy}, \text{ but this is the}$$

Fluxion of the Area AFC , therefore z or $ABD = AFC$.

Ex. 14.

Let AD be the Quadratrix, $AB = x$, $BD = y$, then (by 106.

the Process in Ex. 11. Prob. VIII.) $\dot{x} = \frac{2y\dot{y}}{3a} +$

$$\frac{4y^3\dot{y}}{45a^3} + \frac{4y^5\dot{y}}{315a^5} + \text{Ec.} \text{ Whence } \dot{z} = y\dot{x} = \frac{2y^2\dot{y}}{3a} +$$

$$+ \frac{4y^4\dot{y}}{45a^3} + \frac{4y^6\dot{y}}{315a^5} \text{ Ec. whence } z = \frac{2y^3}{9a} + \frac{4y^5}{225a^3}$$

$$+ \frac{4y^7}{2205a^5} + \text{Ec.}$$

Ex. 15.

Let AD be the Catenary, $AB = x$, $BD = y$ Curve 107.

$AD = v$, the Equation of the Curve is $vv = 2ay + yy$,

whence $v\dot{v} = a\dot{y} + y\dot{y}$, and $v^2\dot{v}^2 = a^2 + y^2 \times \dot{y}^2$, or

$\overline{a+y}^2 \times \dot{v}^2 - a^2\dot{v}^2 = \overline{a+y}^2 \times \dot{v}^2 - \dot{x}^2$, whence $a\dot{v} = a+y \times \dot{x}$; therefore $\dot{z} = y\dot{x} = a\dot{v} - a\dot{x}$, and $z = av - ax$.

Ex. 16.

Let AD be such a mechanical Curve, that the Ordinate 108.

$FD = \text{Arch } AC$ of the Parabola whose Equation is ry

$= uu$; putting $AB = x$, BD or $AF = y$, $FC = u$, Arch

$AC = v$. Then $v = x$, and $\dot{x} = \dot{v} = (\sqrt{y^2 + u^2} =)$

$y^{-\frac{1}{2}}\dot{y} \sqrt{\frac{1}{4}r+y}$, by Prob. VIII. then $\dot{z} = y\dot{x} = yv = y^{\frac{1}{2}}\dot{y}$

$\sqrt{\frac{1}{4}r+y}$. But $v = \text{Fluent of } y^{-\frac{1}{2}}\dot{y} \sqrt{\frac{1}{4}r+y}$, there-

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fore

FIG. fore by Form 11th, $z = \frac{1}{2}r + \frac{1}{2}y \sqrt{\frac{1}{4}ry + yy} - \frac{ry}{16}$.

Ex. 17.

109. Let AD be a Circle, to find the Area ABD , extremely near the Vertex. Let Radius $CA = a$, $AB = x$, $BD = y$, then $z = yx = x \sqrt{2ax - xx}$; but in A , $\sqrt{2ax - xx} = \sqrt{2ax}$; therefore (by Cor. 3. Prop. II.) $z = yx = x \sqrt{2ax}$; and the Area $z = \frac{2}{3}x \sqrt{ax} = \frac{2}{3}xy$, the Area in the very Vertex of the Figure.

COR. After the same Manner in all Curves of a finite Curvature, the Area of the Segment extremely near the Vertex is $\frac{2}{3}$ the Base into the Height.

Ex. 18.

Let the Equation of a Curve be $y = \frac{ax^r + bx^s + dx^t}{fx^v + gx^w}$,
to find the Area of an extremely small Part of the Figure,

when $x = c$. Here $z = yx = \frac{ac^r + bc^s + dc^t}{fc^v + gc^w} \times x$

(by Cor. 3. Prop. II.) therefore $z = \frac{ac^r + bc^s + dc^t}{fc^v + gc^w} \times x$.

Where x is a very small Part of the Abscissa.

COR. Hence the Area of any compound Curve may be nearly found, by finding after this Manner, all the Parts of the Area, belonging to the several small Parts of the Abscissa, and collecting them into one Sum.

Ex. 19.

110. Let BDM be Archimedes' Spiral, $AB = r$, Arch $AC = v$, $BD = y$, by the Nature of the Figure $rv = cy$, whence

whence $\dot{v} = \frac{c\dot{y}}{r}$, and by similar Triangles $\dot{v} = \frac{r\dot{x}}{y}$ F I G.

whence $\dot{x} = \frac{cy}{rr}$; therefore $\dot{z} = \frac{cyy}{2rr}$; and thence

the Area BED or $z = \frac{cy^2}{6rr}$.

Ex. 20.

Let QDC be the reciprocal Spiral, Radius AB=a, 111.
Arch AC=b, Arch AP=v, BD=y; the Nature of the

Curve gives $ab = vy$, whence $v\dot{y} + y\dot{v} = 0$, and $\dot{v} = \frac{-v\dot{y}}{y}$, and by similar Triangles $\dot{v} = \frac{a\dot{x}}{y}$, whence $\dot{x} = \frac{-vy}{a}$, therefore $\dot{z} = \frac{y\dot{x}}{2} = \frac{-vy\dot{y}}{2a} = \frac{-aby}{2a}$

$= \frac{-by}{2}$, and $z = \frac{-by}{2}$; and when corrected, the

Area BCD or $z = b \times \frac{BC - BD}{2}$.

Ex. 21.

Let QDC be the proportional Spiral, whose Nature 111.
is $b\dot{y} = c\dot{x}$, or $\dot{x} = \frac{b\dot{y}}{c}$, therefore $\dot{z} = \frac{y\dot{x}}{2} = \frac{by\dot{y}}{2c}$:

And z or the Area BQD = $\frac{by^2}{4c}$.

Ex. 22.

Let BED be a Kind of Spiral, BD=y, Arch AD 112.
(whose Center is B) = v, and let its Nature be ex-

press'd by this Equation $av^2 = y^3$, or $y = a^{\frac{1}{3}}v^{\frac{2}{3}}$, and

$\dot{y} = \frac{2}{3}a^{\frac{1}{3}}v^{-\frac{1}{3}}\dot{v}$, whence $\dot{z} = v\dot{y} = \frac{2}{3}a^{\frac{1}{3}}v^{\frac{2}{3}}\dot{v}$, and the Area
 $z = \frac{2}{3}a^{\frac{1}{3}}v^{\frac{5}{3}} = \frac{2}{3}vy$ for the Area ABED.

Ex. 23.

Ex. 23.

Let the Equation of a Curve be $a^5 x^2 y^2 - x^9 = a^6 y^3$, put
 $y = \frac{xx}{z}$, and the Equation is transform'd into $a^5 z^2 - x^9 = a^6 \frac{x^2}{z}$

$$-x^9 z^3 = a^6. \text{ Whence } x = \frac{\sqrt[3]{a^5 z - a^6}}{z}, y = \frac{a^5 z - a^6}{z^3}$$

$$\text{and } \dot{x} = \frac{\frac{5}{3} a^5 z \dot{z} - a^6 \dot{z} + a^6 \dot{z}}{z^2 \times a^5 z - a^6} \text{ and } y \dot{x} =$$

$$\frac{\frac{5}{3} a^5 z \dot{z} - a^6 \dot{z} + a^6 \dot{z}}{z^5} = \frac{a^6 \dot{z}}{z^5} - \frac{2 a^5 \dot{z}}{3 z^4}. \text{ And}$$

$$\text{the Fluent} = \frac{-a^6}{4 z^4} + \frac{2 a^5}{9 z^3} = \frac{-a^6 y^4}{4 x^8} + \frac{2 a^5 y^3}{9 x^6} \text{ for the Area.}$$

Ex. 24.

237. Let the Equation of the Curve be $y^4 - 6 a^2 y^2 + 4 x^2 y^2 + a^4 = 0$. By Reduction $x = \frac{\sqrt{-y^4 + 6 a^2 y^2 - a^4}}{2 y}$,

$$\text{let } \frac{y^4 - 2 a^2 y^2 + a^4}{4 y y} = v v, \text{ then } \frac{y y - a a}{2 y} = v, \text{ and}$$

$$\frac{a a - y y}{2 y} = v; \text{ and } a a - v v = a a - \frac{y^4 - 2 a^2 y^2 + a^4}{4 y y}$$

$$= \frac{-y^4 + 6 a a y y - a^4}{4 y y} = x x. \text{ Therefore } x = \sqrt{a a - v v}.$$

$$\text{From the two Values of } v, \text{ we shall have } y y - a a = \pm 2 v y, \text{ and } y = \pm v + \sqrt{a a + v v}, \text{ and } \dot{y} = \pm \dot{v} +$$

$$\frac{v \dot{v}}{\sqrt{a a + v v}}, \text{ then the Fluxion of the Area } x y = \pm v$$

$$\sqrt{a a - v v} + \frac{v \dot{v} \sqrt{a a - v v}}{\sqrt{a a + v v}} = \pm \frac{a a \dot{v} - v^2 \dot{v}}{\sqrt{a a - v v}} +$$

$$\frac{v \dot{v} \sqrt{a a - v v}}{\sqrt{a a + v v}} + \frac{a a \dot{v} - v^2 \dot{v}}{\sqrt{a a - v v}} +$$

a a v v

$\frac{av\dot{v}-v^3\dot{v}}{\sqrt{a^4-v^4}}$; whose Fluent is had from Form 10, of the Table.

Ex. 25.

Let the Equation of the Curve be $x^3+y^3=axy$. 238.

Put $y = \frac{axx}{vv}$, and the Equation is transformed to

$$v^6+x^3a^3=a^2v^4. \text{ Then } x^3=\frac{a^2v^4-v^6}{a^3}, \text{ and } 3x^2\dot{x}$$

$$=\frac{4a^2v^3\dot{v}-6v^5\dot{v}}{a^3}, \text{ and the Fluxion of the Area } y\dot{x}$$

$$=\frac{axx}{vv} \times \frac{4a^2v^3\dot{v}-6v^5\dot{v}}{3x^2a^3} = \frac{4a^2v\dot{v}-6v^3\dot{v}}{3aa}, \text{ whose}$$

$$\text{Fluent is } \frac{2a^2v^2-\frac{3}{2}v^4}{3aa} = \frac{2vv}{3} - \frac{v^4}{2aa} = \frac{2ax^2}{3y} -$$

$$\frac{x^4}{2yy}.$$

Ex. 26.

To find the Quadrature of the Curve defined by the Equation $ax^\lambda+bx^{\mu}y^{\nu}+cy^{\tau}=0$.

Put $y\dot{x}=uz$, and $z=\frac{x^s}{s}$. Then $y=\frac{uz}{x} =$

$$ux^{\frac{s-1}{s}} = us^{\frac{s-1}{s}} z^{\frac{s-1}{s}}, \text{ because } x = \overline{sz}^{\frac{1}{s}}. \text{ Therefore}$$

the Equation is transform'd into this $a\overline{s}^{\frac{\lambda}{s}}z^{\frac{\lambda}{s}}+bu^{\nu}\times$

$$s^{\frac{\mu+\frac{s-1}{s}-1}{s}} \times z^{\frac{\mu+\frac{s-1}{s}-1}{s}} + cu^{\tau}s^{\frac{s-1}{s}} \times z^{\frac{s-1}{s}} = 0. \text{ Di-}$$

vide the Equation by $s^{\frac{\mu+\frac{s-1}{s}-1}{s}} \times z^{\frac{\mu+\frac{s-1}{s}-1}{s}}$, and we

$$\text{shall have } as^{\frac{\lambda-\mu-\frac{s-1}{s}-1}{s}} \times z^{\frac{\lambda-\mu-\frac{s-1}{s}-1}{s}} + bu^{\nu} + cu^{\tau} \times$$

FIG.

$$\frac{s-1-\tau-\nu-\mu}{s} \times z$$

$$\frac{s-1-\tau-\nu-\mu}{s} = 0.$$

Now to make z vanishout of the last Term, put its Index $\frac{s-1-\tau-\nu-\mu}{s} = 0$ then $s = \frac{\mu}{\tau-\nu} + 1$; and $\frac{\lambda-\mu-s-1-\nu}{s} = \frac{\lambda\tau-\mu\tau-\lambda}{\tau-\nu+\mu}$ $= \pi$, by Substitution. Then the Equation comes to this $as^\pi z^\pi + bu^\pi + cu^\pi = 0$, and consequently $sz =$

$$-\frac{b}{a} u^\pi - \frac{c}{a} u^\pi \Big]^{\frac{1}{\pi}}. \text{ Therefore } F:uz = uz - F$$

$$zu = uz - F: \frac{u}{s} \times \left[-\frac{b}{a} u^\pi - \frac{c}{a} u^\pi \right]^{\frac{1}{\pi}}. \text{ But } uz =$$

$$yx; \text{ also } z = \frac{x^s}{s}, \text{ and } uz = \frac{ux^s}{s} = \frac{x}{s} \times ux^{s-1} =$$

$$\frac{xy}{s}. \text{ Therefore}$$

$$F:yx = \frac{xy}{s} - F: \frac{u}{s} \times \left[-\frac{b}{a} u^\pi - \frac{c}{a} u^\pi \right]^{\frac{1}{\pi}}, \text{ the}$$

Area. Hence

$$1. \text{ For Example, let } x^3 - axy + y^3 = 0, \text{ then } F:yx = \frac{2xy}{3} - F: \frac{2}{3} u \times \sqrt[3]{au - u^3}.$$

$$2. \text{ Let } x^5 - x^3y^2 + g^2y^3 = 0, \text{ then } F:yx = \frac{xy}{4} - F: \frac{u}{4 \times uu - gg u^3}.$$

SCHOLIUM.

It may not be improper, in this Place, to insert (from the Transactions) a general Method for determining the Quadratures of Curves, with its Investigation.

Let

Let the Equation of a Curve be $y^m + ax^n + by^p x^q + cy^r x^s + dy^t x^u \mathcal{E}c. = 0$. And its Area $Ayx + By^l x^f + Cy^g x^b + Dy^i x^k + \mathcal{E}c. =$ Fluent of $y\dot{x}$. Put these two Equations into Fluxions, and make the two Values of $\dot{y}$ equal to each other (expunging y^{m-1} by the Equation of the Curve) and we have

$$\frac{anx^{n-1} + bqy^p x^{q-1} + csy^r x^{s-1} + \mathcal{E}c.}{-max^n y^{-1} - \frac{p}{m} by^{p-1} x^q - \frac{r}{m} cy^{r-1} x^s \mathcal{E}c.} =$$

$$\frac{\frac{A}{-1} y + fBy^l x^{f-1} + bCy^g x^{b-1} + \mathcal{E}c.}{Ax + lBy^{l-1} x^f + gCy^{g-1} x^b \mathcal{E}c.} ; \text{ and}$$

multiplying the Numerators and Denominators alternately, there arises,

$$\left. \begin{aligned} anAx^n + bqAy^p x^q &+ csAy^r x^s \\ + anlBy^{l-1} x^{f+n-1} &+ lbqBy^{p+l-1} x^{f+q-1} \\ &+ ganCy^{g-1} x^{b+n-1} \end{aligned} \right\} \mathcal{E}c. =$$

$$\left. \begin{aligned} \frac{1-A}{-1} max^n + \frac{1-A}{-1} \times \frac{m}{-p} by^p x^q &+ \frac{1-A}{-1} \times \frac{r}{-r} cy^r x^s \\ - mafBy^{l-1} x^{f+n-1} &+ \frac{p}{-m} bfBy^{p+l-1} x^{f+q-1} \\ &+ mabCy^{g-1} x^{b+n-1} \end{aligned} \right\} \mathcal{E}c.$$

Now for determining the Indices, compare the homologous Terms. Thus $p=l-1$, whence $l=p+1$. Again $r=p+l-1=g-1$, and thence $r=2p$, and $g=2p+1$. After the same Manner $t=3p$, $i=3p+1$. Put $\ell+n=q$, and then $f=\ell+1$, $s=2\ell+n$, $b=2\ell+1$, $u=3\ell+n$, $k=3\ell+1$, $\mathcal{E}c$. Therefore the Equation of the Curve will be in this Form, $y^m + ax^n + by^p x^{\ell+n} + cy^{2p} x^{2\ell+n} + dy^{3p} x^{3\ell+n} \mathcal{E}c. = 0$, and its Quadrature $Ayx + By^{p+1} x^{\ell+1} + Cy^{2p+1} x^{2\ell+1} + Dy^{3p+1} x^{3\ell+1} \mathcal{E}c. =$ Fluent of $y\dot{x}$.

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Next

Next for determining the Coefficients A, B, C , &c. Put the Coefficients of the homologous Powers of x equal to each other, thus $anA = 1 - A \times ma$,

whence $A = \frac{m}{m+n}$. After the same Manner $B =$

$$\frac{m-p}{+p-m-\ell-n \times A} \left. \vphantom{\frac{m-p}{+p-m-\ell-n \times A}} \right\} \times \frac{b}{am \times \ell+1 + an \times p+1}, C =$$

$$\frac{m-2p \times c}{+2p-2\ell-m-n \times cA} \left. \vphantom{\frac{m-2p \times c}{+2p-2\ell-m-n \times cA}} \right\} \times \frac{1}{a \times m \times 2\ell+1 + n \times 2p+1} : + \frac{m-p \times \ell+1 + \ell+n \times p+1}{\times -bB} \left. \vphantom{\frac{m-p \times \ell+1 + \ell+n \times p+1}{\times -bB}} \right\} \times \frac{1}{a \times m \times 2\ell+1 + n \times 2p+1} : \&c.$$

Hence therefore if $y^m + ax^n + by^p x^{\ell+n} + cy^{2p} x^{2\ell+n} + dy^{3p} x^{3\ell+n} \&c. = 0$, then the Fluent of yx , or the Area of that Curve is,

$$\begin{aligned} & + \frac{m}{m+n} yx. \\ & + \frac{m-p \times b}{+p-m-\ell-n \times Ab} \left. \vphantom{\frac{m-p \times b}{+p-m-\ell-n \times Ab}} \right\} \times \frac{y^{p+1} x^{\ell+1}}{am \times \ell+1 + an \times p+1} \\ & + \frac{m-2p \times c}{+2p-2\ell-m-n \times cA} \left. \vphantom{\frac{m-2p \times c}{+2p-2\ell-m-n \times cA}} \right\} \times \frac{y^{2p+1} x^{2\ell+1}}{am \times 2\ell+1 + an \times 2p+1} \\ & + : \frac{m-p \times \ell+1 + \ell+n \times p+1}{\times -bB} \left. \vphantom{\frac{m-p \times \ell+1 + \ell+n \times p+1}{\times -bB}} \right\} \times \frac{y^{2p+1} x^{2\ell+1}}{am \times 2\ell+1 + an \times 2p+1} \\ & + \frac{m-3p \times d}{+3p-3\ell-m-n \times dA} \left. \vphantom{\frac{m-3p \times d}{+3p-3\ell-m-n \times dA}} \right\} \times \frac{y^{3p+1} x^{3\ell+1}}{am \times 3\ell+1 + an \times 3p+1} \\ & + : \frac{m-2p \times \ell+1 + 2\ell+n \times p+1}{\times -cB} \left. \vphantom{\frac{m-2p \times \ell+1 + 2\ell+n \times p+1}{\times -cB}} \right\} \times \frac{y^{3p+1} x^{3\ell+1}}{am \times 3\ell+1 + an \times 3p+1} \\ & + : \frac{m-p \times 2\ell+1 + \ell+n \times 2p+1}{\times -bC} \left. \vphantom{\frac{m-p \times 2\ell+1 + \ell+n \times 2p+1}{\times -bC}} \right\} \times \frac{y^{3p+1} x^{3\ell+1}}{am \times 3\ell+1 + an \times 3p+1} \\ & + \frac{m-4p \times e}{+4p-4\ell-m-n \times eA} \left. \vphantom{\frac{m-4p \times e}{+4p-4\ell-m-n \times eA}} \right\} \times \frac{y^{4p+1} x^{4\ell+1}}{ma \times 4\ell+1 + na \times 4p+1} \\ & + : \frac{m-3p \times \ell+1 + 3\ell+n \times p+1}{\times -dB} \left. \vphantom{\frac{m-3p \times \ell+1 + 3\ell+n \times p+1}{\times -dB}} \right\} \times \frac{y^{4p+1} x^{4\ell+1}}{ma \times 4\ell+1 + na \times 4p+1} \\ & + : \frac{m-2p \times 2\ell+1 + 2\ell+n \times 2p+1}{\times -cC} \left. \vphantom{\frac{m-2p \times 2\ell+1 + 2\ell+n \times 2p+1}{\times -cC}} \right\} \times \frac{y^{4p+1} x^{4\ell+1}}{ma \times 4\ell+1 + na \times 4p+1} \\ & + : \frac{m-p \times 3\ell+1 + \ell+n \times 3p+1}{\times -bD} \left. \vphantom{\frac{m-p \times 3\ell+1 + \ell+n \times 3p+1}{\times -bD}} \right\} \times \frac{y^{4p+1} x^{4\ell+1}}{ma \times 4\ell+1 + na \times 4p+1} \\ & \&c. \end{aligned}$$

1. Here

1. Here note, $A, B, C, D \&c.$ represent the first, second, third, fourth, $\&c.$ Terms, only leaving out the Powers of x and y .

2. This Series sometimes gives the Quadrature of a Curve in finite Terms; particularly in Trinomials (that is whose Equation consists of three Terms),

when $\frac{\ell - p + m + n}{- \ell m - p n}$ is a positive whole Number; to which add 1, and you have the Number of Terms in the Series.

3. But in Curves of more Terms, there are several Conditions requisite to their exact Quadrability, which it is needless to enumerate, because such Curves seldom admit of an exact Quadrature. It is sufficient to observe, that if $N =$ Number of Terms in the Equation of the Curve, they will sometimes admit of such a

Quadrature when $\frac{N\ell - 2\ell + 2p - Np + m + n}{- \ell m - p n} + 1$

is a positive whole Number, but never else; which Number will then shew the Number of Terms in the Series, that constitutes the Area.

4. When the Quadrature of a Curve is required by this Series; reduce the Equation of the Curve to the preceding Form, and comparing the homologous Terms, the Exponents and Coefficients will be easily determined; which must be substituted into the foregoing general Series as usual. And when any particular Terms are wanting in the given Equation, then the respective Coefficients will be 0, and those Terms of the Series wherein they are found will vanish.

5. And we must first of all enquire whether it will admit of an exact or geometrical Quadrature, and if it will not admit of it in one Form, it may in another. To this Purpose we must divide the whole Equation by some Powers of x and y ; so that in the resulting Equation, there

M m 2

may

may always be one Term without x and another without y ; for this Condition is absolutely necessary to the Equation; and thus you will have a new Form: And this we must do as oft as possible. Now the Number of different Forms any Equation will admit of is $NN - N$, putting N = Number of Terms in the Equation.

6. If it admit of such Quadrature in none of these Forms, try to find the Complement of the Area, by writing x for y and y for x in the Equation of the Curve; and then proceeding with the new Equation in all Respects as before.

EXAMPLE I.

Let $y^3 + x^3 - byx = 0$. Here $m=3$, $n=3$, $p=1$, $\ell = -2$, $a=1$, $b=-b$, and $\frac{\ell - p + m + n}{- \ell m - pn} = 1$; whence the Curve is geometrically quadrable, and the Area $= \frac{1}{2} xy - \frac{y^2 b}{bx}$.

Ex. 2.

Suppose $y^{\frac{3}{2}} - ax^{\frac{1}{2}} + ry^{\frac{1}{2}}x^{\frac{3}{2}} = 0$. Here $m = \frac{3}{2}$, $n = \frac{1}{2}$, $a=-a$, $b=r$, $p=7$, $\ell = \frac{1}{2}$: then $\frac{\ell - p + m + n}{- \ell m - pn} = 1$; and the Area $= \frac{20}{23} xy + \frac{8r}{23a} y^{\frac{5}{2}} x^{\frac{3}{2}}$.

Ex. 3.

Let $y^2 - \frac{4g^2}{b} x^6 + \frac{g}{b} y^2 x^4 = 0$; here the Curve is not quadrable in this Form, therefore divide by y^2 , and then $1 + \frac{g}{b} x^4 - \frac{4gg}{b} y^{-2} x^6 = 0$, where $m=0$,
 $n=4$,

$n=4$, $p=-2$, $\ell=2$, $a=\frac{g}{b}$, $b=-\frac{4gg}{b}$, and

$\frac{-p+m+n}{-em-pn} = 1$; whence the Area $= \frac{2gx^3}{y}$.

Ex. 4.

Let $y^2 - ax^{-1} - 2ay^3x - ay^6x^3 = 0$; here $m=2$,

$n=-1$, $a=-a$, $b=-2a$, $c=-a$, $p=3$, $\ell=2$,

whence $\frac{2\ell-2p+m+n}{-em-pn} = 1$; and the Area of

the Figure is $= 2xy - \frac{2}{3}x^3y^4$: For all the following Terms of the Series will be nothing.

P R O B. XI.

To find any Number of Curves that may be squared.

Let the Abscissa $AB=x$, Ordinate $BD=y$, Area $ABD=z$. Assume any Equation between x and z , this will determine the Area: From that Equation get $\dot{z}$, and substitute its Value in the Equation $\dot{z}=y\dot{x}$, and this will give the Nature of the Curve. FIG. 113.

EXAMPLE I.

Let $x^2=z$, whence $2x\dot{x}=\dot{z}=y\dot{x}$, whence $y=2x$, and the Figure is a Triangle.

Ex. 2.

Let $ax^3=z^2$, and $\dot{z} = \frac{1}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}=y\dot{x}$, whence $\frac{1}{2}\sqrt{ax}=y$, an Equation to a Parabola.

Likewise if $x^3=az$, then $\frac{3x^2\dot{x}}{a}=\dot{z}=y\dot{x}$; whence $3x^2=ay$, an Equation, again, to the Parabola.

Ex. 3.

Ex. 3.

Let $\frac{a^6}{xx} = z^2$, whence $-\frac{a^3}{xx} \dot{x} = \dot{z} = y\dot{x}$; there-
fore $-\frac{a^3}{xx} = y$, whence y being negative, lies on
the other Side of AB .

Ex. 4.

Let $ccaa + ccxx = zz$, or $z = c\sqrt{aa + xx}$, then
 $\dot{z} = \frac{cx\dot{x}}{\sqrt{aa + xx}} = y\dot{x}$; therefore $y = \frac{cx}{\sqrt{aa + xx}}$

Ex. 5.

If $\frac{aa + xx}{b} = z$, then $\dot{z} = \frac{3x\dot{x}}{b} \sqrt{aa + xx} =$
 $y\dot{x}$, and therefore $y = \frac{3x}{b} \sqrt{aa + xx}$.

Ex. 6.

Assume $b - 3xz + \frac{2}{3}z = z^2$, then $\frac{2}{3}\dot{z} - 3x\dot{z} -$
 $3z\dot{x} = 2z\dot{z}$, and $\dot{z} = \frac{3z\dot{x}}{\frac{2}{3} - 3x - 2z} = y\dot{x}$; therefore
 $y = \frac{3z}{\frac{2}{3} - 3x - 2z}$, out of which z may be ex-
punged by Help of the assumed Equation.

Ex. 7.

Let $\overline{e + fx^n}^m = z$, then $\dot{z} = mnfx^{n-1}\dot{x} \times \overline{e + fx^n}^{m-1}$
 $= y\dot{x}$; whence $y = mnfx^{n-1} \times \overline{e + fx^n}^{m-1}$.

P R O B.

P R O B. XII.

Any Curve ABD being given; to find any Number **FIG.**
of Curves ACE whose Areas shall have any
assigned Relation to the Area of the given Curve.

Let ADF be any Curve referred to an Axis, or **114.**
 ADF any mechanical Curve, whose Complement is **115.**
 ABD ; or ADH any Spiral described by the Arch **116.**
 BD , whose Center is A . ACE the Curve required, **117.**
whose Relation to the first is given.

Put $AB = x$, $BD = y$, Area $ABD = z$.

$AC = v$, $CE = u$, Area $ACE = w$.

Assume any two Equations, one of which may contain
the Relation of the Areas z , w ; the other the Relation
of the Abscissa or Ordinate of the given Curve AD
(x or y) to the Abscissa or Ordinate (v or u) of the
other Curve AE . By Help of these two Equations, and
this third $y\dot{x} = \dot{z}$, expunge all Quantities as far as possible,
except v and u , out of the Equation $u\dot{v} = \dot{w}$, and you
will get an Equation for the Nature of the Curve ACE :
And the second assumed Equation will determine the
Quantities of the Abscissas or Ordinates of the two Curves
to have the required Relation.

Ex. 1. Let AD be a Circle, whose Equation is **114.**
 $ax - xx = yy$. Assume $ax = vv$, and $z = w$, then $\dot{x} = **117.**
 $\frac{2v\dot{v}}{a}$; whence $u\dot{v} = \dot{w} = \dot{z} = y\dot{x} = \frac{2v\dot{v}}{a} \sqrt{ax - xx} =$
 $\frac{2v\dot{v}}{a} \sqrt{vv - \frac{v^4}{aa}}$; therefore $u = \frac{2v\dot{v}}{aa} \sqrt{aa - vv}$, the
Equation of the Curve whose Area is equal to that
of the Circle, when $ax = vv$.$

Ex. 2.

FIG. Ex. 2. Let $ax - xx = yy$ as before, and $ax + z =$

114. and $z = w$. Then $u\dot{v} = \dot{w} = \dot{z} = y\dot{x} = \frac{\dot{v} - \dot{z}}{a} y$

117. $\frac{\dot{v} - u\dot{v}}{a} y$: Whence $u = \frac{y - uy}{a}$, or $u = \frac{y}{a + y}$,

Equation to a mechanical Curve.

Ex. 3. Again let $ax - xx = yy$, $cx + z = w$, and $= vv$. Then $u\dot{v} = \dot{w} = c\dot{x} + \dot{z} = c\dot{x} + y\dot{x}$

$\frac{2cv\dot{v}}{a} + \frac{2v\dot{v}}{a} \sqrt{v^2 - \frac{v^4}{aa}}$. Whence $u = \frac{2v\dot{v}}{aa}$

$\sqrt{aa - vv} + \frac{2cv}{a}$.

Ex. 4. Let $ax - xx = yy$ as before, and assume $z = \frac{2y^3}{3a} = w$, and $x = v$. Then $u\dot{v} = \dot{w} = \dot{z} = \frac{2y^2\dot{y}}{a}$

$= y\dot{x} - \frac{2y^2\dot{y}}{a} = \frac{2x\dot{x}}{a} \times \sqrt{ax - xx} = \frac{2v}{a}$

$\sqrt{av - vv}$; hence $u = \frac{2v}{a} \sqrt{av - vv}$.

Ex. 5. Again let $ax - xx = yy$, and $z^2 = w$, $x = v$

Then $u\dot{v} = \dot{w} = 2z\dot{z} = 2zy\dot{x} = 4zyv\dot{v} = 4zv\dot{v}\sqrt{ax -$

$= 4zv\dot{v}\sqrt{av^2 - v^4} = 4zv^2\dot{v}\sqrt{a - v^2}$: Hence $u =$

$4zv\dot{v}\sqrt{a - vv}$, an Equation to a mechanical Curve

98. Ex. 6. Let $cc + xx = yy$ an Equation to an Hyperbola;

117. and assume $z = w$, $xx = cv$. Then $u\dot{v} = \dot{w} = \dot{z} = y\dot{x} =$

$\frac{c\dot{v}}{2\sqrt{cv}} \times \sqrt{cc + cv} = \frac{c\dot{v}}{2v} \sqrt{cv + vv}$. Hence

$u = \frac{c}{2v} \sqrt{cv + vv}$.

Ex. 7. Let $cc + xx = yy$ as before, and $xy - z =$

$xx = w$. Then $u\dot{v} = \dot{w} = x\dot{y} + y\dot{x} - \dot{z} = x\dot{y}$

$\frac{x^2\dot{x}}{\sqrt{cc + xx}} = \frac{cv\dot{v}}{2\sqrt{cv + vv}}$. Hence $u = \frac{cv}{2\sqrt{cv + vv}}$

Ex.

Ex. 8. Let $y = \frac{xx}{\sqrt{ax-xx}}$ an Equation to the F I G.

103.

Gifford; and $\frac{2x}{3}\sqrt{ax-xx} + \frac{1}{3}z = w$, and $x = v$.

117.

$$\text{Then } u\dot{v} + \dot{w} = \frac{2}{3}x \times \sqrt{ax-xx} + \frac{2axx-4x^2\dot{x}}{6\sqrt{ax-xx}} + \frac{1}{3}\dot{z}$$

$$= \frac{ax-xx}{\sqrt{ax-xx}}\dot{x} = \dot{x}\sqrt{ax-xx} = \dot{v}\sqrt{av-vv}.$$

Hence $u = \sqrt{av-vv}$, an Equation to the Circle.

Ex. 9. Let $y = dx^r \times e + fx^n$; assume $z = w$, $v =$

114.

$$e + fx^n)^m, \text{ then will } x = \frac{\sqrt[n]{v^{\frac{1}{m}} - e}}{f}, \text{ and } \dot{x} = \frac{v^{\frac{1}{m}-1}\dot{v}}{mnf^{\frac{1}{n}}}$$

117.

$$\times v^{\frac{1}{m}-1}e, y = dv \times \frac{\sqrt[n]{v^{\frac{1}{m}} - e}}{f}; \text{ whence } u\dot{v} =$$

$$\dot{w} = \dot{z} = y\dot{x} = \frac{dv^{\frac{1}{m}}\dot{v}}{mnf^{\frac{1}{n}}} \times \sqrt[n]{v^{\frac{1}{m}} - e}^{\frac{r+1}{n}-1} : \text{ there-}$$

$$\text{fore } u = \frac{dv^{\frac{1}{m}}}{mnf^{\frac{1}{n}}} \times \sqrt[n]{v^{\frac{1}{m}} - e}^{\frac{r+1}{n}-1}.$$

Ex. 10. Let BD or $y = \text{Arch } AR$ of the Parabo- 118.

la AR , whose Equation is $ax = ss$, putting $AR = p$, 117.

$BR = s$, Tangent $TR = t$; then $\dot{s} = \frac{sp}{t}$. Now as-

$$\text{sume } z = w, y = v, \text{ then will } u\dot{v} = \dot{w} = \dot{z} = y\dot{x} =$$

$$\frac{2vs\dot{s}}{a} = \frac{2vss\dot{p}}{at} = \frac{2vx}{t}\dot{p} = \frac{2vx}{t}\dot{y} = \frac{2vx}{t}\dot{v};$$

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con-

FIG. Ex. 2. Let $ax - xx = yy$ as before, and $ax + z =$

114. and $z = w$. Then $uv = \dot{w} = \dot{z} = y\dot{x} = \frac{\dot{v} - \dot{z}}{a} y$

117. $\frac{\dot{v} - uv}{a} y$: Whence $u = \frac{y - uy}{a}$, or $u = \frac{y}{a + y}$,

Equation to a mechanical Curve.

Ex. 3. Again let $ax - xx = yy$, $cx + z = w$, and $= vv$. Then $uv = \dot{w} = \dot{c}x + \dot{z} = \dot{c}x + y\dot{x}$

$\frac{2cv\dot{v}}{a} + \frac{2v\dot{v}}{a} \sqrt{v^2 - \frac{v^4}{aa}}$. Whence $u = \frac{2v\dot{v}}{aa}$

$\sqrt{aa - vv} + \frac{2cv}{a}$.

Ex. 4. Let $ax - xx = yy$ as before, and assume $z = \frac{2y^3}{3a} = w$, and $x = v$. Then $uv = \dot{w} = \dot{z} = \frac{2y^2\dot{y}}{a}$

$= y\dot{x} = \frac{2y^2\dot{y}}{a} = \frac{2x\dot{x}}{a} \times \sqrt{ax - xx} = \frac{2v\dot{v}}{a}$

$\sqrt{av - vv}$; hence $u = \frac{2v}{a} \sqrt{av - vv}$.

Ex. 5. Again let $ax - xx = yy$, and $z^2 = w$, $x = v$.

Then $uv = \dot{w} = 2z\dot{z} = 2zy\dot{x} = 4zyv\dot{v} = 4zv\dot{v}\sqrt{ax - xx}$

$= 4zv\dot{v}\sqrt{av^2 - v^4} = 4zv^2\dot{v}\sqrt{a - v^2}$: Hence $u = 4zv\dot{v}\sqrt{a - vv}$, an Equation to a mechanical Curve

98. Ex. 6. Let $cc + xx = yy$ an Equation to an Hyperbola; and assume $z = w$, $xx = cv$. Then $uv = \dot{w} = \dot{z} = y\dot{x} = \frac{c\dot{v}}{2\sqrt{cv}} \times \sqrt{cc + cv} = \frac{c\dot{v}}{2v} \sqrt{cv + vv}$. Hence

$u = \frac{c}{2v} \sqrt{cv + vv}$.

Ex. 7. Let $cc + xx = yy$ as before, and $xy - z =$

$xx = cv$. Then $uv = \dot{w} = \dot{xy} + y\dot{x} - \dot{z} = y\dot{x} + \frac{x^2\dot{x}}{\sqrt{cc + xx}} = \frac{cv\dot{v}}{2\sqrt{cv + vv}}$. Hence $u = \frac{cv}{2\sqrt{cv + vv}}$

Ex.

Ex. 8. Let $y = \frac{xx}{\sqrt{ax-xx}}$ an Equation to the F I G.

103.

Gifford; and $\frac{2x}{3}\sqrt{ax-xx} + \frac{1}{3}z = w$, and $x=v$.

117.

$$\text{Then } u\dot{v} + \dot{w} = \frac{1}{3}\dot{x} \times \sqrt{ax-xx} + \frac{2ax\dot{x}-4x^2\dot{x}}{6\sqrt{ax-xx}} + \frac{1}{3}\dot{z}$$

$$= \frac{ax-xx}{\sqrt{ax-xx}}\dot{x} = \dot{x}\sqrt{ax-xx} = \dot{v}\sqrt{av-vv}.$$

Hence $u = \sqrt{av-vv}$, an Equation to the Circle.

Ex. 9. Let $y = dx^r \times e + fx^n$; assume $z=w$, $v=$

114.

117.

$$e + fx^n)^m, \text{ then will } x = \frac{v^{\frac{1}{n}} - e}{f}, \text{ and } \dot{x} = \frac{v^{\frac{1}{n}-1}\dot{v}}{mnf^{\frac{1}{n}}}$$

$$\times v^{\frac{1}{n}-1}, y = dv \times \frac{v^{\frac{1}{n}} - e}{f}; \text{ whence } u\dot{v} =$$

$$\dot{w} = \dot{z} = y\dot{x} = \frac{dv^{\frac{1}{n}}\dot{v}}{mnf^{\frac{1}{n}}} \times v^{\frac{1}{n}-1} : \text{ there-}$$

$$\text{fore } u = \frac{dv^{\frac{1}{n}}}{mnf^{\frac{1}{n}}} \times v^{\frac{1}{n}-1}$$

Ex. 10. Let BD or $y = \text{Arch } AR$ of the Parabo- 118.

la AR , whose Equation is $ax=ss$, putting $AR=p$, 117.

$BR=s$, Tangent $TR=t$; then $s = \frac{sp}{t}$. Now al-

$$\text{sume } z=w, y=v, \text{ then will } u\dot{v} = \dot{w} = \dot{z} = y\dot{x} =$$

$$\frac{2vss}{a} = \frac{2vss\dot{p}}{at} = \frac{2vx}{t}\dot{p} = \frac{2vx}{t}\dot{y} = \frac{2vx}{t}\dot{v};$$

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con-

F I G. consequently $u = \frac{2vx}{t}$, a mechanical Curve.

115. Ex. 11. Let AD be a Cycloid, Diameter of the
117. generating Circle $= o$, $FG = s$. Assume $y = v$, $z = w$,
then by the Property of the Cycloid $yx = sy$; whence
 $uv = w = z = yx = sy = sv$; and therefore $u = s$;
consequently AE is a Circle the same with AG .

Ex. 12. Let AD be a Figure of Arches, where
Arch $AG = AB = x$. Assume $y = v$, $z = w$. And

by the Property of the Circle $\dot{x} = \frac{ay}{2\sqrt{ay - yy}}$

Then $uv = w = z = yx = \frac{avy}{2\sqrt{ay - yy}} = \frac{avv}{2\sqrt{av - vv}}$

Hence $u = \frac{av}{2\sqrt{av - vv}}$.

116. Ex. 13. Let AD be the Spiral of *Archimedes*,
117. whose Equation is $by = xx$; and assume $x = v$, and

$z = w$; then $uv = w = z = yx = yv = \frac{xx}{b} \cdot v = \frac{vv}{b} \cdot v$;

whence $bu = vv$, and AE is a Parabola, convex to-
wards AC .

P R O B. XIII.

To find the Surface of a Solid.

As the Fluxion of any Space is equal to the
describing Line drawn into the Fluxion of the Axis;
so the Fluxion of the Surface of a Solid (generated
by a Line revolving about an Axis) is equal to the
Periphery of that Circle drawn into the Fluxion of
the Line generating that Surface: Therefore

Let

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Let the Abscissa $AP=x$, Ordinate $PM=y$, Curve AM (or BM) $=z$, $c=3.1416 \times 2 =$ the Circumference of the Circle whose Radius is 1; then to find the Surface generated by the Curve AM revolving about the Axis AP ; by the Equation of the Curve expunge one of the indetermined Quantities and its Fluxion out of the Equation $s = cyz$, or $cy\sqrt{x^2 + y^2}$; and find the Fluent.

FIG.

119.

120.

121.

EXAMPLE I.

Let ABD be a right Cone, $AB=a$, $BC=b$, then $y = \frac{bz}{a}$, therefore $s = cyz = \frac{cbz^2}{a}$, and $s = \frac{cbz^2}{2a}$

123.

$= \frac{czy}{2}$. And the Surface of the whole Cone $ABD = \frac{c}{2} \times AB \times BC$.

Ex. 2.

Let AM be a spherical Surface, Radius $=a$, by the Property of the Circle $ax = yz$; therefore $s = cyz = cax$, whence $s = cax$.

119.

Ex. 3.

Let AM be a parabolic Surface, $ay = xx$, then $y = \frac{2xx}{a}$, whence $s = cy\sqrt{x^2 + y^2} = \frac{cx^2x}{aa}\sqrt{aa + 4xx}$:

120.

and the Fluent (corrected) is $s = \frac{caax + 8cx^3}{32aa}\sqrt{aa + 4xx}$

$-\frac{ca^2}{64} \times 2.302585 \text{ Log. } \frac{2x + \sqrt{aa + 4xx}}{a}$, found by Forms the 9th, 11th, and 13th.

Ex. 4.

Let $ax^m = y^{m+1}$ be an Equation to infinite Parabolas.

119.

By the Process in Ex. 4. Prob. VIII. $z = y\sqrt{1 + by^m}$;

N n 2

whence

F I G. whence $s = cy \sqrt{1 + by^{\frac{2}{m}}} = cy^{\frac{1}{m} + 1} y \sqrt{b + y^{-\frac{2}{m}}}$
 119.

Therefore the Fluent of s or $cy \sqrt{1 + by^{\frac{2}{m}}}$ will be had by Form the 15th, when m is the half of any negative odd Number; And the Fluent of

$cy^{\frac{1}{m} + 1} y \sqrt{b + y^{-\frac{2}{m}}}$ will be had also by Form the 15th, when m is any positive whole Number. And

likewise the Fluent of $cy \sqrt{1 + by^{\frac{2}{m}}}$ will be had by Form the 14th when m is the half of an odd Num-

ber; first finding the Fluent of $cy^{\frac{1}{m} - 1} y \sqrt{1 + by^{\frac{2}{m}}}$ by Form the 9th and 13th. In other Cases, Form the 15th or 16th will give the Fluent by infinite Series.

Cor. If $m=1$, then $s = \frac{c}{12a} \times \overline{aa + 4yy}^{\frac{3}{2}} - \frac{cay}{12}$

Ex. 5.

121. In the elliptic Surface BM, whose Center is A, let $AD = a$, $AB = b$, $AP = x$, $PM = y$; then $y =$

$$\sqrt{bb - \frac{bb}{aa} xx}, \text{ and } y = \frac{-bbxx}{a\sqrt{aabb - bbxx}}, \text{ and } z =$$

$$\sqrt{x^2 + y^2} = \frac{bx}{aay} \sqrt{a^4 - aaxx + bbxx} = \frac{bx}{aay} \sqrt{a^4 + ddxx}$$

putting $aa \cap bb = dd$. Therefore $s = cyx = \frac{cbx}{aa}$

$\sqrt{a^4 + ddxx}$. Let $Q = ,017453 \times \text{Degrees in the Arch}$ whose Sine is $\frac{dx}{aa}$, when a is greater than b . Or

$$Q = 2.30258 \text{ Log. } \frac{dx + \sqrt{a^4 + ddxx}}{aa}, \text{ when } a \text{ is less}$$

than b . And we shall have (by Form 10th and 13th, or

by Form the 9th and 13th) $s = \frac{cbaa}{2d} 2 +$

$\sqrt{a^4 + bb - aa x^2}$, for the Surface *BM* revolving about *AP*.

Ex. 6.

In the Hyperboloid *BM*, described by revolving about *AP*, let the Semi-conjugate = *b*, Semi-transvers

124.

AB = *a*, *AP* = *x*, *PM* = *y*; then $y = \sqrt{\frac{bbxx}{aa} - bb}$,

$= \frac{bxx}{a\sqrt{xx - aa}}$, and $\sqrt{x^2 + y^2} = \frac{bx}{aay} \sqrt{ddxx - a^4}$,

putting $dd = aa + bb$. Therefore $s = cy\sqrt{x^2 + y^2} =$

$\sqrt{ddxx - a^4}$; whence (by Form the 9th and 13th)

$= \frac{cbx}{2aa} \sqrt{ddxx - a^4} - \frac{cbaa}{2d} \times 2.30258 \text{ Log.}$

$+ \sqrt{ddxx - a^4}$: And the Fluent corrected,

$= \frac{cbx}{2aa} \sqrt{ddxx - a^4} - \frac{cbb}{2} + \frac{cbaa}{2d} \times 2.30258$

$\log. \frac{da + ba}{dx + \sqrt{ddx^2 - a^4}}$.

Ex. 7.

To find the Surface described by the Hyperbola revolving

124.

about the Conjugate *AC*. Here $x = \frac{a}{b} \sqrt{bb + yy}$, and $\dot{x} =$

$\frac{ayy}{\sqrt{bb + yy}}$, and $\sqrt{x^2 + y^2} = y \sqrt{\frac{bb + \frac{da}{bb} yy}{bb + yy}}$.

The Circumference described by *M* is *cx*, whence $s =$

$\frac{ca}{b} y \sqrt{bb + \frac{da}{bb} yy}$. And (by Form 9 and

13)

F I G

$$13) s = \frac{cay}{bb} \sqrt{bb + \frac{dd}{bb} yy} + \frac{2.302585cabb}{2d}$$

$$\text{Log: } \frac{dy}{bb} + \sqrt{1 + \frac{ddy}{b^2}}$$

Ex. 8.

122.

In the right angled Hyperbola CM, let $AP = PM = y$, and $aa = xy$, and $y = \frac{-yx}{x} = \frac{-aax}{xx}$

whence $\sqrt{x^2 + y^2} = \frac{x}{xx} \sqrt{a^4 + x^4}$, and therefore $s =$

$$cy \sqrt{x^2 + y^2} = \frac{caax}{x^3} \sqrt{a^4 + x^4} = \frac{ca^6x - x^5}{\sqrt{1 + a^4x}}$$

$$+ \frac{caax}{\sqrt{a^4 + x^4}}. \text{ Whence (by Form 3 and 9) } s =$$

$$\frac{caa}{2} \times 2.30258 \times \text{Log. } x^2 + \sqrt{a^4 + x^4} : - \frac{caa}{2xx} \sqrt{a^4 + x^4}$$

But in C, let $x = a$, and $s = 0$, therefore the Fluxion

$$\text{corrected is } s = \frac{caa}{\sqrt{2}} - \frac{caa}{2xx} \sqrt{a^4 + x^4} + \frac{caa}{2}$$

$$2.30258 \text{ Log. } \frac{x^2 + \sqrt{a^4 + x^4}}{aa \times 1 + \sqrt{2}}, \text{ for the Surface described by CM about the Asymptote AP.}$$

Ex. 9.

125.

Let the Ellipsis Mm revolve round the Line AP perpendicular to the Axis CA. Let $CA = d$, $RM = Qm$, then $s = cz \times PM + cz \times Pm = 2cdz$. Whence $s = 2cdz$. And the whole Surface described by RDQ is $cd \times \text{Semi-periphery } RDQ$.

Ex. 10.

126.

To find the Surface of the Ungula of a Cylinder cut off by a Plane FQ. Let F be the Vertex, ADZ the Section

Section made by a Circle perpendicular to the Axis; FIG. 126.
 ED, NI Perpendiculars on the Plane of the Circle
 ADZ; DC, IP Perpendiculars on AZ.

Put $CD=d$, $DF=a$, Radius of the Cylinder $=r$,
 $DI=z$, $CP=x$, then $r-d+PI=\text{Cosine of } z=y$,
 $PI=y+d-r$; and by the Nature of the Circle

$PI=yz$, and by similar Triangles $NI=\frac{a}{d} \times y+d-r$;

$$s=NI \times z = \frac{az}{d} \times y+d-r = \frac{arx}{d} + \frac{az}{d} \times$$

$$r: \text{Whence } s = \frac{arx}{d} + \frac{az}{d} \times d-r; \text{ and}$$

$$\text{the whole Surface } ZFADZ = a \times ADZ - \frac{ar}{d}$$

$$\text{Arch } ZDA - \text{Cord } ZA.$$

Ex. II.

Let the Cissoid NM revolve round the Asymptote
 AP; let $AN=a$, $AP=x$, $PM=y$, then is $x =$

127.

$$\frac{(a-y)^2}{ay-yy} = \frac{(a-y)^{\frac{3}{2}}}{\sqrt{y}}; \text{ whence } \dot{x} = \frac{-a\dot{y}-2y\dot{y}}{2yy}$$

$$ay-yy; \text{ therefore } \dot{z} = \frac{a\dot{y}}{2y} \sqrt{\frac{a+3y}{y}}; \text{ consequent-}$$

$$s = cy\dot{z} = \frac{cay}{2} \sqrt{\frac{a+3y}{y}} = \frac{cay^{-\frac{1}{2}}}{2} \sqrt{a+3y} :$$

whence (by Forms the 9th and 13th) $s =$

$$\frac{1}{4} \sqrt{ay+3yy} + \frac{caa}{4} \times 2.3025 \text{Log. } a+6y+2\sqrt{3ay+9yy} :$$

the Fluent being duly corrected, $s = \frac{1}{2} ca \sqrt{ay+3yy}$

$$+ \frac{2.302585caa}{4} \times \text{Log. } \frac{a+6y+2\sqrt{3ay+9yy}}{a \times 7 + 4\sqrt{3}}$$

Ex. 12.

EX. 12.

FIG. 128. Let DM be the Log. Curve; to find the Surface describes by revolving round the Asymptote AP . Subtangent $PT=a$, $AD=b$, $AP=x$, $PM=y$. the Nature of the Curve $\dot{x} = \frac{ay}{y}$: Whence

$$cy\sqrt{\frac{aay^2}{y^2}} + \dot{y}^2 = cy\sqrt{aa+yy}; \text{ and by Form 9}$$

13) $s = \frac{1}{2}y\sqrt{aa+yy} + \frac{1}{2}caa \times 2.30258 \text{ Log. } \sqrt{aa+yy}$; but in D , $s=0$, and $y=b$; therefore Fluents corrected by Prop. XII. is $s = \frac{1}{2}y\sqrt{aa+yy}$

$$+ \frac{1}{2}b\sqrt{aa+bb} + \frac{1}{2}caa \times 2.30258 \text{ Log. } \frac{y+\sqrt{aa+y^2}}{b+\sqrt{aa+b^2}}$$

EX. 13.

129. Let the Cycloid AM revolve about RG ; Let $AP=PM=y$, Curve $AM=z$. By the Process Ex Prob. VIII. $\dot{x} = \dot{x}\sqrt{\frac{a}{x}}$, therefore $\dot{s} = c \times PR \times \dot{x}$

$$= \frac{a}{x} \times c\dot{x} \sqrt{\frac{a}{x}} = ca^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x} = ca^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}, \text{ whence}$$

$$2ca^{\frac{3}{2}}x^{\frac{1}{2}} - \frac{2}{3}ca^{\frac{1}{2}}x^{\frac{3}{2}} = c\sqrt{ax} \times 2a - \frac{2}{3}x. \text{ And when } x=a, \text{ the whole Surface described by } AMC = \frac{4}{3}ca^2$$

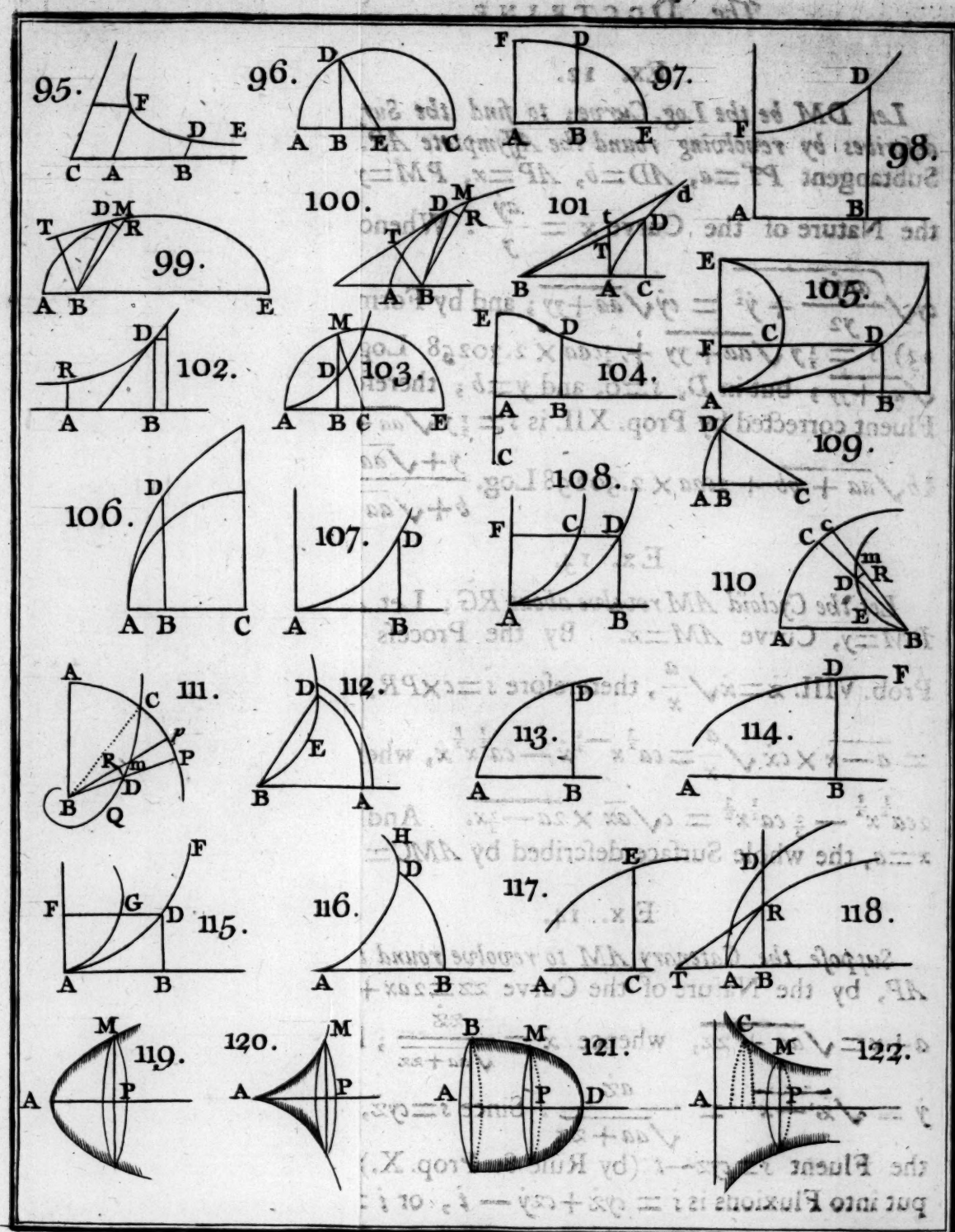
EX. 14.

119. Suppose the Catenary AM to revolve round the AP , by the Nature of the Curve $zz=2ax+xx$,

$$a+x=\sqrt{aa+zz}, \text{ whence } \dot{x} = \frac{z\dot{z}}{\sqrt{aa+zz}}; \text{ likewise}$$

$$\dot{y} = \sqrt{\dot{z}^2 - \dot{x}^2} = \frac{a\dot{z}}{\sqrt{aa+zz}}. \text{ Since } \dot{s} = cy\dot{z}, \text{ afflu}$$

the Fluents $s = cyz - t$ (by Rule 8. Prop. X.) put into Fluxions is $\dot{s} = cy\dot{z} + cxy - \dot{t}$, or $\dot{t} = cy\dot{z}$



$\frac{zz}{+zz}$, and (by Form the 3d) $t = ca\sqrt{aa+zz}$;
 hence $s = cyz - ca\sqrt{aa+zz} = cyz - caa - cax$.
 when corrected $s = cyz - cax$.

PROB. XIV.

To find the Content of solid Bodies.

130.

In any Solid AMm , generated by the Space APM revolving round the Axis AP ; suppose the Plane Mm move along the Axis AP , and by that Motion to describe or generate that Solid; and suppose a given straight line B to move with the same Motion along the Axis AP , and by that Motion to generate a parallelopipedon; the Fluxions of these Solids will be as the describing Planes Mm and B drawn into the Fluxions of their Motions or the Fluxions of the Abscissa AP . Now since $B \times AP =$ Parallelopipedon, $B \times$ Fluxion of $AP =$ its Fluxion, consequently Fluxion of the Solid $AMm =$ describing Plane $Mm \times$ into the Fluxion of AP .

131.

Likewise if the Solid AMB be supposed to be generated by a cylindric Surface $MHbm$ continually extending itself and moving along the Ordinate PM , retaining AP for its Axis; it may be the same proved, that the Fluxion of the Solid $AMHbmA$ generated thereby is $=$ cylindric Surface $MHbm$ multiplied by the Fluxion of the Ordinate PM . Therefore,

To find the Solidity of a Body, let the Abscissa $AP = x$, Ordinate $PM = y$, $MH = u$, $c = 3.1416$ &c. $s =$ solid Content. Then by the Equation of the Curve expunge of the indetermined Quantities (and its Fluxion) of the Equation $\dot{s} = cyy\dot{x}$ for the Solid AMm , or out of

F I G. of the Equation $s = 2cyy$ for the cylindrical Solid AMH or $HMEmb$; and then find the Fluent.

EXAMPLE I.

123. Let ABD be a Cone, Height $AC = a$, CB or $CD = b$, then $bx = ay$; therefore $s = cyyx = \frac{cbbx^2}{aa}$, and $s = \frac{cb^2x^3}{3aa} = \frac{cyyx}{3}$, and the whole Cone = $\frac{cabb}{3} = \text{Base} \times \frac{1}{3} \text{ Height}$.

Ex. 2.

132. Let BF be a Prismoid, whose Bases are right angled Parallelograms, though not similar.

Let its perpendicular Height $AL = b$, $AP = x$, $AD = d$, $BE = s$, $AG = n$, $BC = m$, by similar Triangles $b : x :: s - d : \frac{s-d}{b}x = QI$, and $b : x :: m - n : \frac{m-n}{b}x$

$= IR$; then $KI = d + \frac{s-d}{b}x$, and $IH = n + \frac{m-n}{b}x$; then $s = d + \frac{s-d}{b}x \times n + \frac{m-n}{b}x \times x =$

$dnx + \frac{m-n}{b}dx^2 + \frac{s-d}{b}nxx + \frac{s-d}{b} \times \frac{m-n}{bb}x^2x$;

therefore $s = dnx + \frac{m-n}{2b}dx^2 + \frac{s-d}{2b}nxx + \frac{s-d \times m-n}{3bb}x^3$ for the Solid $HIKF$. And when $x = b$,

then the whole Solid $BF = dm + sn + 2sm + 2dn : \times \frac{b}{6} = : s + \frac{1}{2}d \times m + \frac{1}{2}d + \frac{1}{2}s \times n : \times \frac{1}{3}b$.

Ex. 3.

130. Let AMm be the Segment of a Sphere, its Diameter $= a$, then $yy = ax - xx$, therefore $s = cyyx = caxx - cx^2x$.

Therefore $s = \frac{cax^2}{2} - \frac{cx^3}{3}$, for the Segment $AMPm$. And when $x = a$, the whole Sphere $= \frac{ca^3}{6} = \frac{2}{3}$ the circumscribing Cylinder.

Ex. 4.

Let AMm be the Segment of a Sphere of an exceeding small Height, here $yy = ax$ nearly, therefore (by Cor. 3.

Prop. II.) $\dot{s} = cyy\dot{x} = cax\dot{x}$; therefore $s = \frac{caxx}{2} = \frac{cxy^2}{2}$, nearly.

Hence in any Solid of a finite Curvature, the Content of a Segment of a very small Height, will be found to be half the Base drawn into the Height.

Ex. 5.

Let the Solid be a Paraboloid, where $ax = yy$; then $\dot{s} = cyy\dot{x} = cax\dot{x}$; therefore $s = \frac{caxx}{2} = \frac{cyyx}{2} = \frac{1}{2}$ Base $\times$ Height.

And in general if $x = y^m$, then $\dot{x} = my^{m-1}\dot{y}$; and then $\dot{s} = cyy\dot{x} = cmy^{m+1}\dot{y}$; and therefore $s = \frac{cm}{m+2}y^{m+2}$

$= \frac{cm}{m+2}y^2x$. Therefore in the conic Parabola,

where $m = \frac{1}{2}$, that is when the Curve is convex towards AP , then the Solidity $= \frac{cyyx}{5} = \frac{1}{5}$ Base $\times$ Height.

Ex. 6.

Let AmD be a Parabolic Spindle, generated by the Parabola AMD (whose Axis is DT) revolving round the Ordinate AT .

Let $AT = b$, $DT = d$, $DH = v$, $MH = u$, then MP or $y = d - v$, and $av = uu$ by the Nature of the Curve:

O o 2

Whence

FIG.

Whence $y = -v = \frac{-2ui}{a}$; wherefore $s = 2cuy =$

$$\frac{-4cu^2y}{a} = \frac{-4cu^2u}{a} \times a = \frac{-4cu^3}{a}. \text{ Therefore the Flu-}$$

ent $s = \frac{-4cd u^3}{3a} + \frac{4cu^3}{5aa}$; but in A where s is 0,

$u=b$; therefore the Fluent corrected is $s = \frac{4cdb^3}{3a}$

$$- \frac{4cd u^3}{3a} + \frac{4cu^3}{5aa} = \frac{4cb^3}{5aa} = \frac{8cdb^3}{15a} - \frac{4cu^3}{15a} \times$$

$5d - 3v$, for the Solid $AMHbm$; and when v and

$u=0$, the whole Solid $ADd = \frac{8cdb^3}{15a} = \frac{8cbdd}{15} =$

$\frac{8}{15}$ the Base $\times$ Height.

Ex. 7.

130.

Let AMm be an Hyperboloid, Transverse $= a$, Conju-

gate $= b$. Then $yy = \frac{bb}{aa} \times ax + xx$; and $s = cyyx$

$$= \frac{cbbx}{aa} \times ax + xx. \text{ And the Fluent } s = \frac{cbbxx}{2a} +$$

$$\frac{cbbx^3}{3aa} = (\text{expunging } bb) \frac{3ax + 2xx}{6a + 6x} \times cyy = \frac{3a + 2x}{6a + 6x}$$

$\times$ Base $\times$ Height.

Hence if $x=0$, then the Solid $= \frac{1}{2}$ Base $\times$ Height, and if x be infinite, then the Solid $= \frac{1}{3}$ Base $\times$ Height: Therefore the Hyperboloid is always between $\frac{1}{2}$ and $\frac{1}{3}$ the circumscribing Cylinder, and is nearly $= \frac{5}{12}$ thereof.

Ex. 8.

133.

Let the Hyperbola CM revolve round the Conjugate AP , A being the Center, $AC=b$, Semiconjugate $= a$.

By the Nature of the Figure $yy = bb + \frac{bb}{aa} xx$; whence

whence $\dot{s} = cyy\dot{x} = cbb\dot{x} + \frac{cbbx^2\dot{x}}{aa}$, and $s = cbbx +$

$\frac{cbb}{3aa}x^3 = \frac{2}{3}cbbx + \frac{1}{3}cyyx$, the Solid described by *ACMP*.

Ex. 9.

Let the Hyperbola *DM* revolve round the Asymptote *AP*, let $AB = b$, $BD = d$, $xy = aa$. Then $\dot{s} = cyy\dot{x}$

$= \frac{ca^4\dot{x}}{xx}$, whence $s = \frac{ca^4}{x}$, let the Solid begin at

B; and then the Fluent corrected is $s = \frac{ca^4}{b} - \frac{ca^4}{x}$

$= caa \times d - y$.

Note if *CAB* is not a right Angle, s must be diminished in the Ratio of the Radius to the Sine of that Angle. When $y = 0$, the infinitely long Solid $DMPB = caad = cddb = \text{Base} \times AB$.

Or thus: Let $AP = x$, $PM = y$, $aa = xy$, then $u = x = \frac{aa}{y}$, whence $\dot{s} = 2cuy\dot{y} = 2caay\dot{y}$; therefore $s = 2caay$
 $= 2cyyx$, for the infinitely long Solid *AHMQP*.

Ex. 10.

Let *DM* the Log. Curve revolve round the Asymptote *AP*, let Subtangent $TP = a$, then $y\dot{x} = ay\dot{y}$; and $\dot{s} =$

$cyy\dot{x} = cayy\dot{y}$; therefore $s = \frac{cayy}{2} = \text{Base} \times \frac{1}{2}TP$, for the infinitely long Solid *MPAD*.

Ex. 11.

Let *AM* be a Spheroid, *A* the Center, $AE = a$, AC

$= b$, then $yy = bb - \frac{bb}{aa}xx$; whence $\dot{s} = cyy\dot{x} = cbb\dot{x}$

$-\frac{cbbx^2\dot{x}}{aa}$; therefore $s = cbbx - \frac{cbb}{3aa}x^3 = \frac{2}{3}cbbx$

$+\frac{1}{3}cyyx$. And when $x = a$, then $s = \frac{2}{3}cbba = \frac{2}{3}$
 $\text{Base} \times \text{Height}$.

Ex.

FIG.

EX. 12.

136.

Let the Segment $ADMN$ of a Circle revolve about the Sine AN ; $CD=r$, $CA=b$, AP or $CQ=x$, $PM=y=\sqrt{rr-xx}-b$; then $\dot{s} = cyy\dot{x} = c\dot{x} \times rr + bb - xx - 2bcx\sqrt{rr-xx}$; and the Fluent $s = cx \times rr + bb - \frac{1}{2}xx - 2bc \times \text{Area } CDMQ$ the Solid described by $ADMP$.

EX. 13.

137.

In the Cissoid AM revolving round $AR=a$, we have

$$y = \frac{xx}{\sqrt{ax-xx}}, \text{ whence } \dot{s} = cyy\dot{x} = \frac{cx^3\dot{x}}{a-x} = -cx^2\dot{x}$$

$$-cax\dot{x} - caa\dot{x} + \frac{ca^3\dot{x}}{a-x}: \text{ Therefore (by Form 4 \& c)}$$

$$s = -\frac{cx^3}{3} - \frac{cax^2}{2} - caax - 2.30258ca^3 \times \text{Log.}$$

$$\frac{a-x}{a-x}: \text{ And when duly corrected } s = -\frac{1}{3}cx^3 - \frac{1}{2}cax^2 - caax + 2.30258ca^3 \text{ Log. } a - 2.30258ca^3 \text{ Log. } a -$$

$$x = 2.302585ca^3 \times \text{Log. } \frac{a}{a-x} - \frac{1}{3}cx^3 - \frac{1}{2}cax^2 - caax.$$

EX. 14.

138.

Let the Cissoid revolve round the Affymptote AP , then

$$x = \frac{a-y^{\frac{3}{2}}}{\sqrt{y}} = u: \text{ Whence } \dot{s} = 2cuy\dot{y} = 2cxy\dot{y} =$$

$$2cy^{\frac{1}{2}}\dot{y} \times a - y^{\frac{3}{2}}: \text{ Therefore (by Forms 10, 11 and 13)}$$

$$s = ca \times \text{Segment } ACD + \frac{2}{3}c \times ay - yy^{\frac{3}{2}}, \text{ for the infinitely long Solid } ADMQ. \text{ And the whole } ABMQ = ca \times \text{Semicircle } ACB.$$

EX. 15.

139.

Let the Chonchoid DM revolve about the Affymptote AP ;

$$AD=a, AC=b, \text{ then } b+y\sqrt{aa-yy}=xy, \text{ and } u=x \text{ therefore}$$

therefore $\dot{s} = 2cuy\dot{y} = 2cxy\dot{y} = 2cy \times \overline{b+y} \sqrt{aa-yy} = \text{FIG,}$
 $2cy\sqrt{aa-yy} + 2cy\dot{y}\sqrt{aa-yy}$. Describe the Quadrant
 MDK; then (by Forms 10 and 13 &c) $s = 2cb \times$
 Area AKLR $= \frac{2}{3}c \times \overline{aa-yy}^{\frac{3}{2}}$. And, duly corrected,
 $s = 2cb \times \text{Area AKLR} + \frac{2}{3}ca^3 - \frac{2}{3}c \times \overline{aa-yy}^{\frac{3}{2}}$, for
 the infinitely long Solid ARMQP.

Ex. 16.

Let the Conchoid DM revolve round the Axis DR, 139.
 where the Circle described by RM = cxx, and the
 Fluxion of DR or a-y is -y; therefore the Fluxion

of the Solid $\dot{s} = -cxy\dot{y} = -cy \times \frac{\overline{b+y}^2}{yy} \times \overline{aa-yy} =$

$-cy \times \overline{b+y}^2 - \frac{aabbcy}{yy} - \frac{2aabc}{y} \dot{y} - aacy$: Whence

$\dot{s} = \frac{c}{3} \times \overline{b+y}^3 + \frac{cbbaa}{y} - 2aabc \times 2.302585 \times$

Log. y: -aacy. And by due Correction (when y=a)

$\dot{s} = \frac{1}{3}c \times \overline{b+y}^3 - \frac{1}{3}c \times \overline{b+a}^3 + \frac{aabbcc}{y} - abbc + a^3c$

$-aacy + 2.30258 \times 2caab \times \text{Log. } \frac{a}{y}$.

Ex. 17.

Let the Cycloid AMG revolve about the Axis RG, 129.

Let AR=a, AP=x, Rp=u, pM=y, Arch AB=v,

PB=z, then x=a-y, and $\dot{x} = -\dot{y}$, and by the Na-

ture of the Figure $u = v + z$, and $\dot{u} = \dot{v} + \dot{z} =$

$\frac{a\dot{x}}{2\sqrt{ay-yy}} + \frac{2y-a}{2\sqrt{ay-yy}} \dot{x} = \dot{x} \sqrt{\frac{y}{a-y}} = -\dot{y} \sqrt{\frac{y}{a-y}}$:

then $\dot{s} = 2cuy\dot{y}$; and assume $s = cuyy + t$, this put

into Fluxions we shall find $\dot{t} = -cyy\dot{u} = \frac{cy^{\frac{5}{2}}\dot{y}}{\sqrt{a-y}}$;

whence

F I G. whence by Forms the 10th and 11th) $t = \frac{5ca^3}{16} \phi -$
 $\frac{8yy + 10ay + 15aa}{24} \times c \sqrt{ay - yy}$ (putting $\phi =$
 $\frac{8 \text{ Sectors } CBR}{aa}$) : Therefore $s = cyy + \frac{5}{2} ca \times$
 $\text{Sector } CBR - \frac{15caa + 10acy + 8cyy}{24} \sqrt{ay - yy}$, for
 the Solid described by $PMGR$. And when $y = a$
 then $s = \frac{5}{2} ca \times \text{Semicircle } ABR$, the Solid described
 by the whole Space $AMGR$ revolving about RG .

Ex. 18.

130. Let the Catenary AM revolve about the Axis AP ,
 here $AP = x$, $PM = y$, $AM = z$, and $zz = 2ax + xx$
 whence (see Ex. 8. Prob. III.) $zy = ax$. Now $\dot{s} =$
 $cyy\dot{x}$, and assume $s = cyyx + t$, this in Fluxions gives
 $\dot{t} = -2cxy\dot{y} = (\text{because } \dot{y} = \frac{a\dot{x}}{z}) - 2cay \times \frac{x\dot{x}}{z} =$
 $(\text{because } x\dot{x} = z\dot{z} - a\dot{x}) - 2cay \times \dot{z} - \frac{a\dot{x}}{z} = -2cay$
 $\times \dot{z} - \dot{y} = 2cay\dot{y} - 2cay\dot{z}$; then assume $t = cyy -$
 $2cayz + u$, this Equation in Fluxions will produce
 $\dot{u} = 2caz\dot{y} = 2caax$, whence $u = 2caax$. Therefore the
 Solid $s = cxyy + cyy - 2cayz + 2caax$.

Ex. 19.

~~139.~~
~~148~~
 130 Let the Abscissa $AP = x$, Ordinate $PM = y$, $p = 3.1416$
 to find the Solidity of a Body described by the Revolution
 of any Curve defined by this Equation $yy = A + Bx + Cx^2$.

The Fluxion of the Solid $pyy\dot{x} = A + Bx + Cx^2 : \times$
 $p\dot{x}$, and the Fluent is $A + \frac{Bx}{2} + \frac{Cx^2}{3} : \times px$ for
 the Solidity; out of this expunge either $\frac{Bx}{2}$ or
 $\frac{Cx^2}{3}$ by the Equation of the Curve, and you will

have

have the Solid $= \overline{yy + A - \frac{1}{3}Cxx} : \times \frac{px}{2} =$

$\overline{yy + 2A + \frac{1}{2}Bx} : \times \frac{px}{3}$, which contains two general Theorems for all Solids of this Kind. Therefore from the Equation of the Curve get the Values of A, B, C , and substitute them in that Expression of the Solid where you have the fewest of the Quantities A, B, C , as in the following Cases.

1. Let $y = ax$ denote a Cone, then $yy = aaxx$, and you have $A = 0, B = 0, C = aa$. Whence the Solid $= yy$

$$\times \frac{px}{3}.$$

2. In the Frustrum of a Cone, $b =$ Radius of the Top, and $y - b = ax$, or $yy = bb + 2abx + aaxx$; here $A = bb, B = 2ab, C = aa$, and the Solid $= \overline{yy + 2bb + bax}$

$$\times \frac{px}{3} = (\text{expunging } ax) \overline{yy + by + bb} \times \frac{px}{3}.$$

3. For the Segment of a Sphere whose Radius is $r, yy = 2rx - xx$. Here $A = 0, B = 2r, C = -1$, and

$$\text{the Solid} = \overline{yy + \frac{1}{3}xx} \times \frac{px}{2} = \overline{yy + rx} \times \frac{px}{3}.$$

4. For the Frustrum of the Sphere, $yy = rr - xx$; here $A = rr, B = 0, C = -1$, and then the Frustrum $=$

$$\overline{yy + 2rr} \times \frac{px}{3}.$$

5. In the Spheroid, $a =$ Transverse, $d =$ Conjugate,

$$\text{and } yy = \frac{ddx}{a} - \frac{ddxx}{aa}, \text{ here } A = 0, B = \frac{dd}{a},$$

$$C = \frac{-dd}{aa}. \text{ Solid} = \overline{yy + \frac{ddx}{2a}} \times \frac{px}{3}, \text{ for the Segment.}$$

6. For the middle Frustrum of the Spheroid, $2t =$ Transverse, $2c =$ Conjugate, then $yy = cc - \frac{cc}{tt} xx$,

P p

where

F I G.

130.

where $A = cc$, $B = 0$, $C = \frac{-cc}{tt}$, the Solid =

$$\overline{yy + 2cc} \times \frac{px}{3}.$$

7. In a Paraboloid $yy = ax$, here $A = 0$, $B = a$, $C = 0$
the Solid = $\overline{yy} \times \frac{px}{2}$.

8. For the Frustrum of a Paraboloid, $b =$ Radius
of the lesser Base, $yy = bb + ax$; here $A = bb$, $B = a$,
 $C = 0$, then the Solid = $\overline{yy + bb} \times \frac{px}{2}$.

9. In the Hyperboloid, $2a =$ Transverse, $2d =$ Con-
jugate, and $yy = \frac{2ddx}{a} + \frac{dd}{aa}xx$; here $A = 0$, $B =$
 $\frac{2dd}{a}$, $C = \frac{dd}{aa}$. Solid = $\overline{yy + \frac{ddx}{a}} \times \frac{px}{3}$, the
Segment.

P R O B. XV.

140. *The Nature of the reflecting Curve AMn, and the luma-
nous Point L being given; To find the Focus F, or the
Concourse of the nearest reflected Rays MF, nF.*

Take the Particle of the Curve Mn infinitely
small, and let C be the Center and CM the Radius of
Curvature of the Arch Mn ; and on ML , nL , ML ,
 nF let fall the Perpendiculars CE , Ce , CG , Cg ; also
on the Centers L , F , describe the small Arches Me ,
 no ; then the little Triangles Mon , Mnr are equal and
similar, and $Mo = nr$. By the Nature of Reflection
the Angle $LMC = CMF$, and $LnC = CnF$, whence
 $CE = CG$, and $Ce = Cg$. Now if $CE = Ce$ (that is,
 L falls in E or e) then will $CG = Cg$; that is, the Point

F will fall in G or g when M and n coincide: But if F I G .
 Ce be less than CE , that is, if L falls below E , then 140.

will Cg be less than CG , and the Intersection F will
 then fall above G towards M ; and the contrary.

The Triangles LEQ , LMr ; and Fon , FSG are
 similar, and $EQ = CE - Ce = CG - Cg = SG$; and

$Mr = no$, and $FG = MG - MF = ME - MF$;

therefore $LM : LE :: (Mr : EQ :: no : SG ::$

$FM : FG ::) FM : ME - MF$. Whence $MF =$

$$\frac{LM \times ME}{$$

$$2LM - ME.$$

Otherwise thus: Draw the Tangent MP , and the
 Perpendicular LP , and let $LM = y$, $LP = u$, $ME = v$,

and by Prob. V. $MC = \frac{yy}{u}$, whence by similar Tri-

angles $vy = \frac{yuy}{u}$, or $vu = uy$, therefore $MF =$

$$\frac{uyy}{2yu - uy}.$$

1. Wherefore if CM be the Radius of Curvature,
 CE perpendicular to LM , and LP perpendicular to the
 Tangent PM , and we make the Distance of the radiat-
 ing Point $LM = y$, $ME = v$, $LP = u$: Then compute

the Value of v by Prob. V. and take $MF = \frac{vy}{2y - v}$:

And when AM is convex towards L write $-v$ instead
 of $+v$.

2. Or find u from the Nature of the Curve, by Help of

which expunge it out of the Equation $MF = \frac{uyy}{2yu - uy}$.

COR. The Curve FfH passing through all the

Points F , or touching all the reflected Rays MF , mf

is called the Catacaustic or Caustic by Reflexion. In

which any Portion HfF of the Curve is $= LM + MF$

$- LA - AH$. For drawing mL infinitely near ML ,

and Mo , mr perpendicular to mF , ML ; then since

$Mo = Mr$, therefore $Lm + mf = LM + Mf$, or $LM +$

Mf

F I G. $Mf - Lm - mf = 0$; and adding fF , $LM + MF - Lm - mf = fF$; but these Moments are as the Fluxions, and therefore the Fluents thereof will be equal, that is the Curve $HF = LM + MF - LA - AH$.

EXAMPLE I.

142. Let AM be a right Line, then v is infinite, whence

$$MF = \frac{vy}{2y-v} = \frac{vy}{-v} = -y.$$

Or thus: u is a standing Quantity and $\dot{u} = 0$, therefore $MF = \frac{uy\dot{y}}{2y\dot{u} - u\dot{y}} = \frac{uy\dot{y}}{-u\dot{y}} = -y$: Whence Perpendicular $PF = PL$, and F is the Focus of the reflected Rays.

Ex. 2.

143. Let MD be a Circle, C the Center; then $MF = \frac{vy}{2y-v}$. And when y is infinite, $MF = \frac{1}{2}v = \frac{1}{2}ME$.

And if MD be very small, $MF = \frac{LD \times DC}{2LC + CD}$.

And if $LC = CD$, then $MF = \frac{1}{3}v = \frac{1}{3}ME = \frac{1}{3}LM$.

Ex. 3.

144. Let MD be a Parabola, and let the Rays be parallel to the Axis DB , then y is infinite, whence $MF =$

$$\frac{vy}{2y-v} = \frac{1}{2}v. \text{ But by Prob. V. } v = \frac{2 \cdot M\mathcal{Q}^2}{r}$$

(putting $r =$ Latus Rectum) also $\mathcal{Q}T = \frac{2 \mathcal{Q}M^2}{r}$,

therefore $\mathcal{Q}T = v$, and $\mathcal{Q}F = \frac{1}{2}v = MF$, and therefore F is the Focus of the Parabola.

Ex. 4.

145. Let DM be a Parabola, and let all the Rays be perpendicular to the Axis $D\mathcal{Q}$. Let $DP = x$, $PM = z$, $M\mathcal{Q} = s$, and $rx = zz$: Then by Prob. V. ME or

$= \frac{4ssz}{rr}$, and $ss = zz + \frac{1}{4}rr$ by the Nature of the Fi-

gure: Whence $MF = \frac{vy}{2y-v} = \frac{1}{2}v = \frac{2z^2}{rr} + \frac{1}{2}z = \frac{2zx}{r} + \frac{1}{2}z$.

Ex. 5.

Let DM be the Log. Spiral, the Center L the luminous Point. By Prob. V. $v=y$, whence $MF = \frac{vy}{2y-v} = y$. 146.

Ex. 6.

Suppose DM to be an Ellipsis, L the Focus, the Transverse $= 2r$, Conjugate $= 2c$; then by the Pro- 147.

perty of the Curve PL or $u = \frac{cy}{\sqrt{2ry-yy}}$, and $\dot{u}$

$= \frac{cry\dot{y}}{(2ry-yy)^{\frac{3}{2}}}$; whence $MF = \frac{uy\dot{y}}{2y\dot{u} - u\dot{y}} =$

$\frac{cy^2\dot{y} \times 2ry - yy}{cy^3\dot{y}} = 2r - y$; consequently (since the

Angle $LMP = FMT$) the Point F is the other Focus.

And in like Manner it will be found that Rays issuing from one Focus of an Hyperbola, and reflected by the Curve will diverge from the other Focus.

P R O B. XVI.

The Nature of the refracting Curve AMn , and the luminous Point L being given; to find the Focus F , or the Point where the nearest refracted Rays MF , nF concur; and where they meet the Axis of the Figure. 148.

Suppose the Arch Mn to be infinitely small, and let C be the Center and CM the Radius of Curvature in

F I G. in M ; and let fall the Perpendiculars CE , Ce on the Rays of Incidence LM , Ln , and the Perpendiculars CG , Cg on the refracted Rays MF , nF ; and let the Sine of Incidence CE to the Sine of Refraction Cg be as m to n . On the Centers L , F , describe the small Arches Mr , Mo . Now since Ce exceeds CE therefore Cg exceeds CG , they being in a given Ratio; whence MF , nF intersect beyond G .

The Figures $GCME$ and $onMr$ are similar; therefore $ME : MG :: Mr : Mo = \frac{MG \times Mr}{ME}$; and LM

$: LQ :: Mr : Qe = \frac{LQ \times Mr}{LM}$: And by the Property of Refraction, $m : n :: (Ce : Cg :: CE : CG :$

$Ce - CE : Cg - CG ::) eQ : Sg = \frac{n \times LQ \times Mr}{m \times LM}$

and by the similar Triangles FoM , Fgs ; $Mo - Sg$

$Mo :: MG : MF = \frac{m \times MG^2 \times LM}{m \times MG \times LM - n \times LQ \times ME}$

149. Produce MF till it intersect the Axis of the Curve in O , and let $LA = d$, $AO = f$, $LM = y$, $MO =$

$AH = x$, $MH = z$, then $y = \sqrt{z^2 + d + x^2}$, and $s =$

$\sqrt{zz + f - x^2}$. And it is $y' : -s' :: rn : on :: m : n$

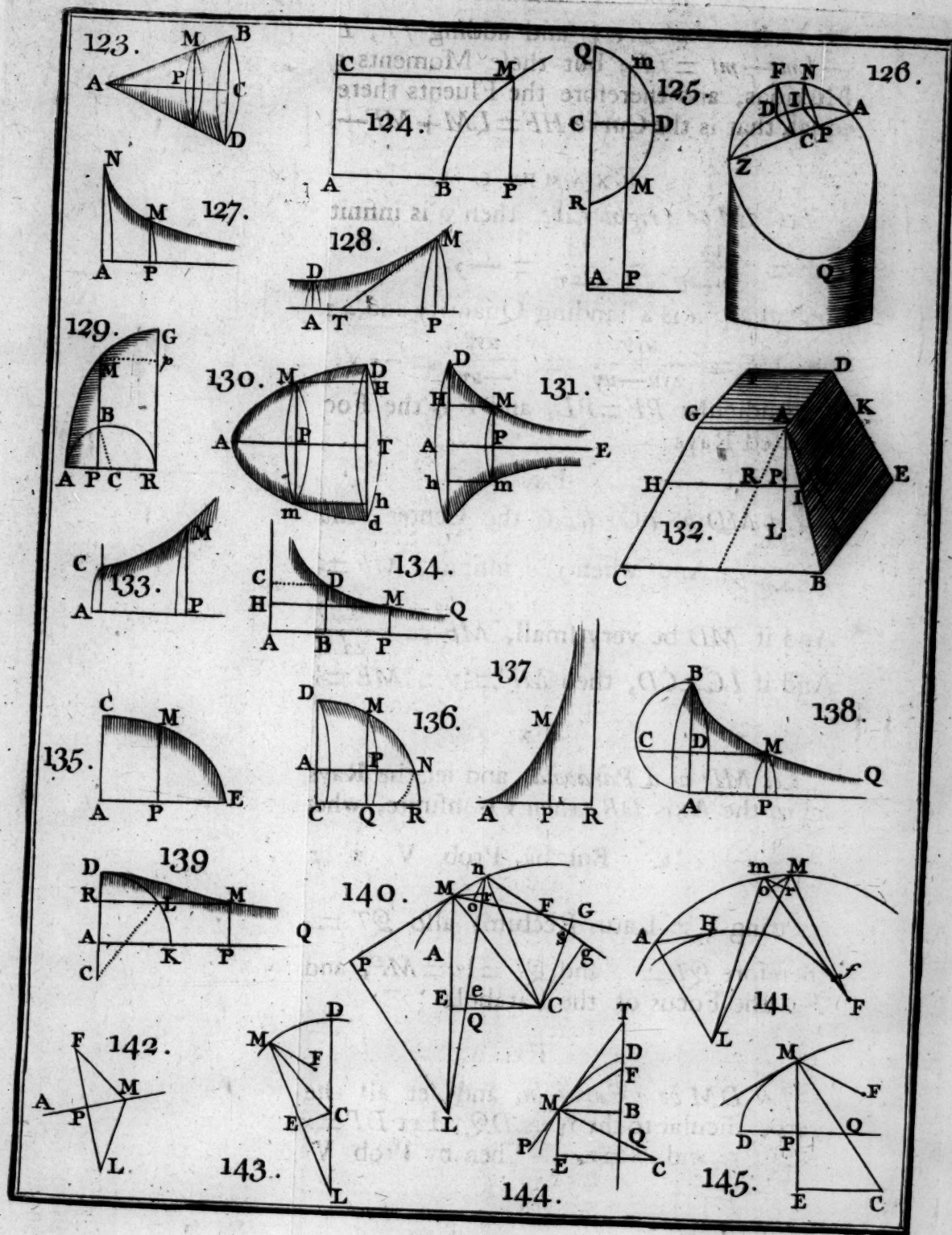
and $ny' = -ms'$ or $ny = -ms$, that is $\frac{nz\dot{z} + nd\dot{x} + nx\dot{x}}{\sqrt{zz + d + x^2}}$

$= \frac{mf\dot{x} - mxx\dot{x} - mzz\dot{z}}{\sqrt{zz + f - x^2}}$. Therefore,

149. 1. To find the Focus F , let CM be the Radius of Curvature, CE perpendicular to LM , and CG to MG , $ME = v$, $MG = u$. $LM = y$, m and n the Sines of Incidence and Refraction. Then find v and u by Prob. V.

and take $MF = \frac{muuy}{muy - nuv - nvy}$. If AM is con-

cave



the Rays converge when they fall on the Curve *AM*, write $-v$ for v , and $-u$ for u : And the Rays converge when they fall on the Curve *AM*, write $-y$ for y .

To find the Point *O* where the refracted Ray meets the Axis of the Curve, let $LA = d$, $AO = f$, $AH = x$, $HO = z$, then by the Nature of the Curve expunge

or z out of the Equation $\frac{nz\dot{z} + nd\dot{x} + nx\dot{x}}{\sqrt{zx + d + x^2}} =$

$\frac{-m\dot{x} - mz\dot{z}}{\sqrt{zx + f - x^2}}$; and by Reduction find f : And

if the Curve is concave towards *L*, write $-x$, for x , &c.

COR. The Curve *NfF* passing through all the Points *F*, or which touches all the refracted Rays *KN*, *F*, is called the Diacaustic or Caustic by Refraction.

And any Portion of it $NF = FM + \frac{n}{m}ML - NK -$

KL . For supposing Mn infinitely small, and draw-

Mo, *Mr*, Perpendiculars on *fn*, *nL*; then by

the Nature of Refraction $m : n :: rn : on = \frac{n}{m}rn$,

therefore $on = \frac{n}{m}rn = 0$; that is $M\dot{f} - nf -$

$\times \overline{Ln} - \overline{LM} = 0$, and adding *Ff*; we have $F\dot{f} =$

$F\dot{f} - nf - \frac{n}{m} \times \overline{Ln} - \overline{LM}$; but these Moments are

the Fluxions; whence the Fluents will be equal, or

$N = FM - KN - \frac{n}{m} \times \overline{LK} - \overline{LM}$.

Ex.

EXAMPLE I.

FIG. Let AM be a plane Surface, then v, u are infinite,

151. whence $\frac{muuy}{muy - nvv - nvy} = \frac{muuy}{-nvv}$. Now let $r =$ the infinite Radius of Curvature, the Perpendicular $LA = p$, then by the similar Triangles LAM , MCE , $v = \frac{rp}{y}$; also $mm \times rr - uu = nn \times rr - vv$, by the right angled Triangles MEC , MGC : Whence $uu = \frac{mm - nn}{mm} rr + \frac{nn}{m^2} vv$; therefore $MF = \frac{-muuy}{nvv} = -\frac{mm - nn}{mnrvv} \times rry - \frac{ny}{m} = -\frac{m^2 - n^2}{mnp^2} y^3 - \frac{n}{m} y$. And near the Point A , $Af = -\frac{m}{n} p$.

Ex. 2.

152. Let parallel Rays fall on the convex Side of the Sphere AM ; then y is infinite, and $MF = \frac{muuy}{muy - nvv - nvy} = \frac{muu}{mu - nv}$, and near the Vertex A , $u = v = AC$; whence $Af = \frac{m \times AC}{m - n}$.

Ex. 3.

153. Let parallel Rays fall on the concave Side of the Sphere AM ; then y is infinite, and v, u are negative; whence $MF = \frac{muuy}{-muy - nvv + nvy} = \frac{muu}{nv - mu}$, and at the Vertex A , $v = u = AC$, then $AF = \frac{mu}{n - m} = \frac{m}{n - m} AC$.

Ex.

Ex. 4.

Suppose the Rays proceeding from L, to fall on the convex FIG.

Side of the Sphere AM; then $MF = \frac{muuy}{muy - nvv - nvy}$. 154.

And in or near the Vertex, $MF = \frac{muy}{m - n \times y - nu}$

$$\frac{mry}{m - n \times y - nr}$$

And to find where the refracted Ray meets the Axis of the Sphere, $zz = 2rx - xx$, and $z\dot{z} = r\dot{x} - x\dot{x}$, by which expunge z and $\dot{z}$ out of the Equation

$$\frac{nzz + nd\dot{x} + nxx}{\sqrt{zz + d + x^2}} = \frac{mf\dot{x} - mxx - mzz}{\sqrt{zz + f - x^2}}, \text{ and there}$$

$$\text{arises } \frac{nr + nd}{\sqrt{2rx + 2dx + dd}} = \frac{mf - mr}{\sqrt{2rx - 2fx + ff}}, \text{ by}$$

Reduction of which f is found. And in the Vertex

$$\text{A where } x \text{ is } 0, \frac{mf - mr}{f} = \frac{nr + nd}{d}, \text{ or } f =$$

$$\frac{mdr}{m - n \times d - nr}$$

Ex. 5.

If Rays fall on the concave Surface of the Sphere AM, 155.

then $MF = \frac{muuy}{-muy - nvv + nvy}$. And near the Ver-

$$\text{tex, } Mf = \frac{muy}{n - m \times y - nu} = \frac{mry}{n - m \times y - nr}$$

To find where MF cuts the Axis, here $zz = 2rx - xx$; put the Equation into Fluxions, and write $-x$ for x , $-\dot{x}$ for $\dot{x}$, $-r$ for r in the Equation

$$\frac{nzz + nd\dot{x} + nxx}{\sqrt{zz + d + x^2}} = \frac{mf\dot{x} - mxx - mzz}{\sqrt{zz + f - x^2}}, \text{ and we}$$

Qq

have

F I G.

have $\frac{nr - nd}{\sqrt{dd - 2dx + 2rx}} = \frac{-mr - mf}{\sqrt{ff + 2fx + 2rx}}$; and

in the Vertex $\frac{nr - nd}{d} = \frac{-mr - mf}{f}$, or $f =$
 $\frac{mrd}{n - m \times d - nr}$

155.

COR. Hence we may find if the Sphere in this Case has a geometrical Focus, or such a one where all Rays are refracted accurately to one Point. When this happens, AO must be a determined Quantity for all Parts of the Curve, the same as at the Vertex; and therefore the Powers of x must vanish out of the general Equation expressing AO ; consequently the Coefficients of each Power, being equated, must destroy one another. Thus the Equation for AO be-

ing $n^2 \times r - d^2 \times \sqrt{ff + 2fx + 2rx} = m^2 \times r + f^2 \times \sqrt{dd - 2dx + 2rx}$, then putting $n^2 \times r - d^2 \times 2f + 2rx = m^2 \times r + f^2 \times 2r - 2d \times x$, after reducing the Equation, there will be found $n^2 d = n^2 r - m^2 r - m^2 f = n^2 r - m^2 r - m^2 \times \frac{mrd}{nd - md - nr}$; from whence will be

found $d = \frac{m+n}{n}r$, therefore $f = -\frac{m+n}{m}r$, which

being negative, shows that the Focus O lies on the same side as the radiating Point L . Therefore when the Distance of the radiating Point is such that $d = \frac{m+n}{n}r$, then F or O will be a geometrical Focus for all Rays falling on all Points of the Sphere; and d will be to f : as m to n , and the Rays after Refraction will diverge from O .

And on the contrary Rays converging to O , and falling on the Sphere, will all be accurately refracted to their Focus L .

Ex.

Ex. 6.

Let parallel Rays fall upon the Spheroid *AM*, let *F I G.*
Transverse = *b*, Parameter = *a*, then $zz = ax -$ 156.

$\frac{a}{b}xx$, and $zx = \frac{1}{2}ax - \frac{a}{b}xx$; and since *d* is infinite,
therefore the Equation for *AO* becomes $n =$
 $mf - mx - \frac{ma}{2} + \frac{ma}{b}x$, whence *f* will be found;

$\sqrt{ax - \frac{a}{b}xx + f - x}$,
and in the Vertex where $x=0$, $mf - \frac{1}{2}ma = nf$, or
 $f = \frac{\frac{1}{2}ma}{m-n}$.

COR. And to find if the Spheroid has a geometrical
Focus, we have from the above Equation, $n^2 \times : ff$ 156.

$-2fx + xx$
 $+ ax - \frac{a}{b}xx = mf - \frac{ma}{2} + 2mf - ma \times \frac{max}{b} - mx +$
 $\frac{ma}{b} - m$ x^2 , then equalling the Coefficients of *x* and

x^2 , $n^2 \times a - 2f = 2mf - ma \times \frac{ma}{b} - m$, also mn

$\frac{-ma}{b} = \frac{ma}{b} - m$. From the former Equation

$2m^2 - 2n^2 - \frac{2m^2a}{b} \times f$, or $2m^2 - 2n^2 - \frac{2m^2a}{b} \times$

$\frac{\frac{1}{2}ma}{m-n} = m^2a - \frac{m^2a^2}{b} - n^2a$, hence we find $a =$

$\frac{m^2 - n^2}{m^2}b$. And from the second Equation, we get

likewise $a = \frac{m^2 - n^2}{m^2}b$; therefore we may conclude

that the Spheroid has a geometrical Focus, when $a =$

$\frac{b \times m^2 - n^2}{m^2}$; and then $f = \frac{\frac{1}{2}ma}{m-n} = \frac{m+n}{2m}b =$

156.

$\frac{1}{2}b + \frac{bn}{2m}$; but $\frac{bn}{2m} =$ Distance of the Focus from the Center; therefore $f =$ Distance of the remoter Focus from the Vertex. And therefore in this Case all parallel Rays falling on all Points of the Spheroid, will be accurately refracted to the further Focus of the Figure.

And on the contrary, Rays issuing from the further Focus of this Spheroid and refracted at the Surface, will all emerge parallel to the Axis.

P R O B. XVII.

F I G. To find the Center of Gravity of any Line, Surface or Solid.

The Center of Gravity of a Body is that Point upon which, if it were suspended, it would rest in any given Position.

157.

Let MN be any Figure or Solid Body, regular or irregular; C its Center of Gravity; and suppose it to be suspended in C upon the horizontal Line SC , and the Point of Suspension to be in S . Let all the infinitely small Particles of the Body be reduced to the Line SC , situated respectively in Planes perpendicular to SC ; as at e, d, b , &c. Now by Mechanics, the Force of any Particle e to turn the Body about C , is as its Weight and Distance from C , that is, as $Ce \times e$, or as $SC - Se \times e$, for the same Reason the Force of all the particles between S and C will be $SC - Se \times e + SC - Sd \times d + SC - Sb \times b$, &c. In like Manner the Force of a Particle beyond C , as g , to turn the Body the contrary Way about C , is $Cg \times g$, or

Sg

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F I G.

157.

$SC \times g$; and the Force of all the Particles g, b, i ,

will be $Sg - SC \times g + Sb - SC \times b + Si - SC \times i$,

But as C is the Center of Gravity, the Forces

on each Side will be equal; that is, $SC - Se \times e +$

$SC - Sd \times d$, &c. = $Sg - SC \times g + Sb - SC \times b$, &c.

and $SC \times e + SC \times d + SC \times g + SC \times b$, &c. =

$Se \times e + Sd \times d + Sg \times g + Sb \times b$, &c. whence $SC =$

$\frac{Se \times e + Sd \times d + Sb \times b + Sa \times a + Sg \times g + Sh \times h}{e + d + b + a + g + h}$, &c.

$$= MN.$$

Now if any one of the variable Distances as Sb be

called x ; the Body MN , S ; then will $Sb \times b = xs'$;

and the Sum of all the $Sa \times a + Sb \times b + Sd \times d$, &c.

= Sum of all the xs , or the Fluent of xs ; and the

Sum of all the $a + b + d$, &c. = Sum of all the s , or

the Fluent of s ; that is the Body MN ; therefore

$$SC = \frac{\text{Fluent of } xs}{\text{Fluent of } s} = \frac{\text{Fluent of } xs}{\text{Body } MN}.$$

To find the Center of Gravity; let s = Line, Surface

or Solid: Multiply the Fluxion of the Line, Surface or

Solid (s) by the Distance (of the Center of Gravity of

the generating Point, Line or Plane) from the Axis of

Suspension; and find the Fluent z ; then $\frac{z}{s}$ = Distance

of the Center of Gravity from the Point of Suspension.

EXAMPLE I.

Let SB be a right Line or Cylinder, S the Point of

Suspension; $SB = x$; then $\dot{x} = \dot{x}$, and $z = \frac{1}{2}xx$;

therefore $\frac{z}{s} = \frac{1}{2}x$, for the Distance of the Center

of Gravity, SC .

Ex.

Ex. 2.

- FIG. 159. In the Triangle SQD , whose Point of Suspension is S , let SF bisect the opposite Side QD , then the Centre C is in the Line SF ; draw AE parallel, and SG perpendicular to QD , put $SF=a$, $SB=x$, $SI=v$, $SG=b$, $QD=b$, $AE=y$. By similar Triangles $v = \frac{bx}{a}$, and $\dot{v} = \frac{bx}{a}$, $y = \frac{bx}{a}$; then $\dot{z} = xy\dot{v} = \frac{bbx^2\dot{x}}{aa}$ and $z = \frac{bbx^3}{3aa}$. Also $\dot{s} = y\dot{v} = \frac{bbx\dot{x}}{aa}$, and $s = \frac{bbx^2}{2aa}$: Therefore $\frac{z}{s} = \frac{2}{3}x$, and when $x=a$, $\frac{z}{s} = \frac{2}{3}a = SC$.

Ex. 3.

160. Let AM be the Arch of a Circle, s its Center, $AS=b$, $SE=r$, $SB=x$, $AM=v$, $BM=y$. It is evident the Center of Gravity of any Arch AEG is in the Line SE that bisects it. Whence $\dot{z} = x\dot{v} =$ (by the Nature of the Circle) $-ry$; and $z = -ry$; and the Fluent corrected is $z = br - ry$. Whence $\frac{z}{s} = \frac{rb - ry}{v}$, and when $y=0$, $\frac{z}{s} = \frac{rb}{v} = SC$, the Distance of the Center of Gravity of the Arch AEG from S .

Ex. 4.

161. For the Sector of a Circle MmS , whose Center and Point of Suspension is S ; let Arch $Mm=c$, Radius $SM=r$, $Mm=a$, $SQ=x$, $QDq=v$. Then $Qq = \frac{ax}{r}$, and by the last Example, the Distance of the Center of Gravity of the Arch QDq from S is $= \frac{ax^2}{rv}$, therefore $\dot{z} = \frac{ax^2\dot{x}}{r}$, and $z = \frac{ax^3}{3r}$; there-

fore

$$\frac{z}{s} = \frac{ax^3}{3r \times \frac{1}{2}xv} = \frac{2ax^2}{3rv} = \frac{2ax}{3c}; \text{ and}$$

$$\text{when } x=r, \text{ then } SC = \frac{2ar}{3c}.$$

Ex. 5.

For the Circular Area PQD , Let SD or $SE=r$,

$QD=b$, $PQ=c$, $SB=x$, $PA=v$, $AB=y$, by the Na-

ture of the Circle $y = \sqrt{rr-xx}$; then $\dot{z} = yxx =$

$xx\sqrt{rr-xx}$, and $z = -\frac{1}{3} \times \overline{rr-xx}^{\frac{3}{2}}$; corrected $z =$

$$\frac{rr-bb}{3} - \frac{rr-xx}{3} = \frac{c^3-y^3}{3}.$$

Also $s = \text{Area}$

$$PABQ = \frac{vr+xy-cb}{2}; \text{ whence } \frac{z}{s} = \frac{2}{3} \times \frac{c^3-y^3}{vr+xy-cb}$$

$= Sc$. And the Distance of the Center of Gravity

of the whole $PADQ$ from S is $= \frac{2c^3}{3vr-3cb}$.

Again in Respect of the Axis of Suspension SD ;

since the Center of Gravity of the describing Line y ,

is in the Middle of BA , therefore $\dot{z} = \frac{1}{2}y \times y\dot{x} =$

$$\frac{rrx-x^2\dot{x}}{2}, \text{ whence } z = \frac{r^2x}{2} - \frac{x^3}{3}.$$

But (in Q ,

$x=b$, $z=0$) the Fluent corrected is $z =$

$$\frac{3rrx-3rrb-x^3+b^3}{6}, \text{ therefore } \frac{z}{s} = \frac{2}{3} \times \frac{3r^2x-3r^2b-x^3+b^3}{vr+xy-cb};$$

and when $x=r$,

$$\frac{2r^3-3r^2b+b^3}{3rv-3cb} = \text{Distance of}$$

the Center of Gravity of the Semi-segment $PADQ$

from QD .

Ex. 6.

In the Parabola $rx=yy$, let $SP=x$, $PM=y$, then

in Respect of the Axis of Suspension ST , $\dot{z} = yxx =$

$$xx\sqrt{rx}; \text{ and } z = \frac{2}{3}xx\sqrt{rx}; \text{ And } s = \frac{2}{3}xy = \frac{2}{3}x\sqrt{rx};$$

therefore

162.

163.

FIG. therefore $\frac{z}{s} = \frac{1}{3}x$ the Distance of the Center of Gravity from ST .

Again in Regard to the Axis of Suspension S because the Center of Gravity of the describing Line is in the Middle of MP , therefore $\dot{z} = \frac{1}{2}yy\dot{x} = \frac{y^2}{r}$ and thence $z = \frac{y^4}{4r}$; and $s = \frac{2y^3}{3r}$; Therefore $\frac{z}{s} = \frac{1}{3}y$, the Distance from SP .

Ex. 7:

164. For the hyperbolic Area $BCMP$, between the asymptotes. Let $SB = b$, $BC = c$, $SP = x$, $PM = cb = xy$. Then in Regard to the Axis SD , $\dot{z} = yx\dot{x} = cb\dot{x}$, and $z = cbx$, but in B , $x = b$, therefore by Correction $z = cb \times \overline{x - b}$. And $\frac{z}{s} = \frac{cb \times \overline{x - b}}{\text{Area } BCMP}$

Again for the Axis of Suspension SP , $\dot{z} = \frac{ccbb}{2xx} \dot{x}$, and $z = \frac{-ccbb}{2x} = \frac{-cby}{2}$; corrected $= \frac{cb}{2} \times \overline{c - y}$ and $\frac{z}{s} = \frac{cb \times \overline{c - y}}{2 \text{ Area } BCMP}$.

Ex. 8.

165. Let AMB be an Ellipsis, S the Center, $AS = a$, $SB = b$, $SQ = y$, $QM = x = \frac{a}{b} \sqrt{bb - yy}$, then for the elliptic Space, $\dot{z} = xy\dot{y} = \frac{ayy}{b} \sqrt{bb - yy}$; and $= -\frac{a}{3b} \times \overline{bb - yy}^{\frac{3}{2}}$, corrected $z = \frac{abb}{3} - \frac{a}{3b} \times \overline{bb - yy}^{\frac{3}{2}}$; Thence $\frac{z}{s} = \frac{ab^2 - a \times \overline{bb - yy}^{\frac{3}{2}}}{3b \times \text{Area } SAMQ}$.

Likewise

Likewise for the Distance from SB , $\dot{z} = \dot{xxx}y = F I G.$

$$\frac{aay}{2} - \frac{aay^2y}{2bb}, \text{ and } \mathcal{Q} = \frac{aay}{2} - \frac{aay^3}{6bb}; \text{ whence}$$

$$\frac{z}{s} = \frac{3bbaay - aay^3}{6bb \times \text{Area } SAMQ}.$$

Ex. 9.

Let SMP be the hyperbolic Space, Transverse $= 2a$, 163.

$$\text{Conjugate} = 2b, SP = x, PM = y = \frac{b}{a} \sqrt{2ax + xx},$$

$$\text{whence } xx = 2aa + \frac{aa}{bb} yy - \frac{2aa}{b} \sqrt{bb + yy}, \text{ and } \dot{xx} =$$

$$\frac{aa}{bb} \dot{yy} - \frac{aay\dot{y}}{b\sqrt{bb + yy}}; \text{ whence } \dot{z} = y\dot{xx} = \frac{aa}{bb} y^2\dot{y}$$

$$- \frac{aay^2\dot{y}}{b\sqrt{bb + yy}}, \text{ and } z = \frac{aay^3}{3bb} - \frac{aay}{2b} \sqrt{bb + yy} +$$

$$\frac{aab}{2} \times 2.30258 \text{ Log. } y + \sqrt{bb + yy}, \text{ whence } \frac{z}{s} =$$

$$\frac{z}{\text{Area } SMP} = \text{Distance from } ST.$$

Then for the Distance from SP , we have $\dot{z} =$

$$\frac{yy\dot{x}}{2} = \frac{bb\dot{x}}{2aa} \times \sqrt{2ax + xx}, \text{ and the Fluent } z = \frac{bbxx}{2a}$$

$$+ \frac{bbx^3}{6aa}; \text{ therefore } \frac{z}{s} = \frac{3abbxx + bbx^3}{6aa \times \text{Area } SPM}.$$

Ex. 10.

For the Surface of a right Cone, let $SD = f$, $c =$ 166.

Circumference of the Base, Axis $SB = d$, $SM = v$,

$SF = x$. Then it is plain its Center of Gravity is in

the Axis SB . $\frac{cx}{d} =$ Circumference of the Circle

MQ ; and by similar Triangles $v = \frac{fx}{d}$, and $\dot{v} =$

R r

$f\dot{x}$

FIG. $\frac{f\dot{x}}{d}$; therefore $\dot{z} = \frac{cx^2\dot{x}}{d} = \frac{cfx^2\dot{x}}{dd}$; and $z = \frac{cfx^3}{3dd}$
 also $s = \frac{cx^2}{2d} = \frac{cfx^2}{2dd}$: Therefore $\frac{z}{s} = \frac{1}{2}x = SC$.

Ex. 11.

166. For the Cone (or Pyramid) SDE, let the Base = b
 the rest as in the last Example; then the Circle MG
 $= \frac{bxx}{dd}$; and $\dot{z} = \frac{bx^2\dot{x}}{dd}$, and $z = \frac{bx^3}{3dd}$; also
 $= \frac{bx^3}{3dd}$: Therefore $\frac{z}{s} = \frac{1}{2}x = SC$ the Distance of
 the Center of Gravity from S.

Ex. 12.

167. Let SMD be a Sphere, $SP = x$, $PM = y$, Radius
 $= r$, $c = 3.1416$, then $y = \sqrt{2rx - xx}$, and $\dot{z} = cyx\dot{x}$
 $= 2crx\dot{x} - cx^2\dot{x}$; therefore $z = \frac{2crx^2}{3} - \frac{cx^3}{4}$
 and $s = crx^2 - \frac{cx^3}{3}$ (by Ex. 3. Prob. XIV.); there-
 fore $\frac{z}{s} = \frac{8rx - 3xx}{12r - 4x}$ for the Distance of the Cen-
 ter of Gravity from S.

Ex. 13.

167. For the Spheroid SMD, whose Center is C, let $SC =$
 a , $CF = b$, $SP = x$, $PM = y$, $c = 3.1416$, then $yy =$
 $\frac{bb}{aa} \times 2ax - xx$; and $\dot{z} = cyx\dot{x} = \frac{cbb}{aa} \times 2ax^2 - x^3$
 and $z = \frac{cbb}{aa} \times \frac{2ax^3 - \frac{1}{4}x^4}{4}$; and $s = \frac{cbb}{aa} \times \frac{ax^2 - \frac{1}{4}x^3}{4}$
 and $\frac{z}{s} = \frac{8ax - 3xx}{12a - 4x}$.

Ex. 14.

To find the Center of Gravity of the Solid SBDm, FIG. generated by a partial Revolution of the Parabola SMD. 168. about the Axis SB.

Let S be the Point of Suspension, let $SB = d$, $BD = b$, $SP = x$, $PM = y$, $ax = yy$, Arch $Dd = c$; then Arch $Mm = \frac{cy}{b}$; therefore $\dot{z} = \frac{cyy}{2b} \dot{x} = \frac{cax^2 \dot{x}}{2b}$; and $z = \frac{cax^3}{6b}$: and $s = \frac{cax^2}{4b}$, therefore $\frac{z}{s} = \frac{2}{3}x$, the Distance from ST.

Again for its Distance from SB, let Chord $Dd = f$, then Chord $Mm = \frac{fy}{b}$; and by Ex. 4. the Distance of the Center of Gravity of the Sector PMm from $P = \frac{2fy}{3c}$; therefore $\dot{z} = \frac{fy^3 \dot{x}}{3b} = \frac{2fy^4 y}{3ab}$; and $z = \frac{2fy^5}{15ab}$; therefore $\frac{z}{s} = \frac{8fy^5}{15caax^2} = \frac{8fy}{15c}$, the Distance of the Center of Gravity from SP, and in the Plane that passes through the Axis and bisects the Base.

Ex. 15.

Let the Hyperbola CM revolve round the Asymptote SP, and describe an Hyperboloid CMB: Let $SB = b$, $BC = d$, $SP = x$, $PM = y$, $c = 3$, 1416. $bd = xy$; and

$$\dot{z} = cyy\dot{x} = \frac{bbddc\dot{x}}{x}, \text{ whence } z = bbddc \text{ Log. } x:$$

And corrected $z = bbddc \text{ Log. } \frac{x}{b}$: Also $\dot{s} = cyy\dot{x} =$

$$\frac{cbbdd\dot{x}}{xx}; \text{ and } s = -\frac{cbbdd}{x}, \text{ and corrected } s =$$

$$cbbdd \times \frac{x-b}{bx}; \text{ therefore } \frac{z}{s} = \frac{bx}{x-b} \times \text{Log. } \frac{x}{b}.$$

F I G. = Distance of the Center of Gravity of the Solid from *SA*.

Ex. 16.

163. Let the Solid be an Hyperboloid, Transverse = $2a$, Conjugate = $2b$, $c = 3.1416$, $SP = x$, $PM = y$, = $\frac{b}{a} \sqrt{2ax + xx}$; whence $\dot{z} = cyy\dot{x} = \frac{cbb\dot{x}}{aa} \times \frac{2ax^2 + x^3}{2ax + xx}$; whence $z = \frac{cbb}{aa} \times \frac{\frac{2}{3}ax^3 + \frac{1}{4}x^4}{2ax + xx}$; also $\dot{s} = \frac{cbb}{aa} \dot{x} \times \frac{2ax + xx}{2ax + xx}$; and $s = \frac{cbb}{aa} \times \frac{axx + \frac{1}{2}x^2}{2ax + xx}$; therefore $\frac{z}{s} = \frac{8ax + 3xx}{12a + 4x}$ = Distance of the Center of Gravity from the Vertex *S*.

P R O B. XVIII.

To find the Centers of Percussion and Oscillation.

The Center of Percussion is that Point in the Axis of a vibrating Body, which striking against an immovable Obstacle, the Body shall incline to neither Side, but rest as it were in Equilibrio, on that Point.

And the Center of Oscillation is the Point in the Axis of a vibrating Body, in which if a small Body or Particle be placed, it shall perform its Vibrations after the same Manner, in the same Time, and with the same angular Velocity as the whole Body.

170. To find the Center of Percussion; through the Point of Suspension *C*, and Center of Gravity, draw the Axis of the Body *CO*; and suppose *O* to be the Center of Percussion; through *CO* draw the Plane in which the Center of Gravity moves, and imagine the Body to be divided into innumerable small Prisms, all

all perpendicular to this Plane, and let them be supposed to be reduced to, or situated in, the Points where they intersect this Plane; and let p be one of these small Prisms. Draw pf perpendicular to CO , and pd perpendicular to Cp ; then pd will be the Direction of p 's Motion as it revolves about C ; and the Body being stopt at O , p will urge the Point d forward, with a Force proportional to its Magnitude and Velocity, that is as $p \times Cp$; therefore the Force where-with p acts at d in a Direction perpendicular to CO , will be $p \times Cf$. And the Force by which p endeavours to turn the Body about O , will be as $p \times Cf \times dO$ or $p \times Cf \times CO - Cd$, that is as $p \times Cf \times CO - p \times Cp^2$. Now since the Sum of all these Forces to turn the Body about O must be $= 0$, therefore all the $p \times Cf \times CO - p \times Cp^2 = 0$, or all the $p \times Cf \times CO =$ all the $p \times Cp^2$; therefore $CO = \frac{\text{Sum of all the } p \times Cp^2}{\text{Sum of all the } p \times Cf}$.

For the Center of Oscillation. Through the Centers of Motion C and of Gravity G draw the Axis CO , and let O be the Center of Oscillation. Draw the horizontal Line Cr , and Or , Gg , pn perpendicular thereto, and pf perpendicular to CO .

By Reason of the equal angular Velocities of all the Particles of the Body; the absolute Motion of any Particle q (and consequently the Force that generates it) will be as $Cq \times q$; and therefore a Force acting at

that can generate that Motion in q , is as $\frac{Cq}{Cn} \times Cq$

$\times q$, or $\frac{Cq^2 \times q}{Cn}$, by the Power of the Leaver: Let

$\Sigma =$ Sum of all the $Cq^2 \times q$ in the Body, and let this be as (the Weight of) the Particle p . If the Weight of the Particle p , acting at n , generates any Motion

in the whole Body AD , then $\frac{Cq^2 \times q}{s} p =$ that Part

of the Gravity of p which generates the Motion of

171.

the

F I G. the Particle q ; and the Force acting at q , and the

171.

Motion of q generated by that Force, is $\frac{Cn}{Cq}$

$\frac{Cq^2 \times q}{s}$ or $\frac{Cn \times Cq \times q}{s}$ and the Velocity of q is

$\frac{Cn \times Cq \times p}{s}$, and its angular Velocity $\frac{Cn \times p}{s}$. After

the same Manner any other Force $\frac{Cq^2 \times q}{s}$ acting

at n , will generate the same angular Velocity in

any other Particle q ; and consequently the Sum of

all the Forces $\frac{Cq^2 \times q}{s}$ or the Weight p will generate

the same angular Velocity in all the Particles q together or in the whole Body. Now since the Weight of

any Particle p will generate an angular Velocity in the

Body AD about C , which is as $\frac{Cn \times p}{\text{Sum of all the } Cp^2 \times p}$

therefore the angular Velocity which all the Particles

p can generate is as $\frac{\text{the Sum of all the } Cn \times p}{\text{Sum of all the } Cp^2 \times p}$. In

like Manner the angular Velocity which the Gravity

of a Particle p placed in O would generate in itself is

as $\frac{Cr \times p}{CO^2 \times p}$ or as $\frac{Cr}{CO^2}$. But because of the Equality of the Vibrations and correspondent Accelerations, this last must be equal to the Sum of the former; whence

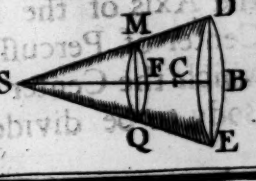
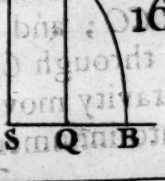
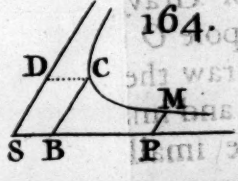
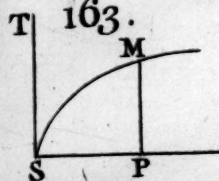
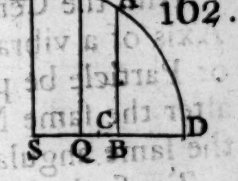
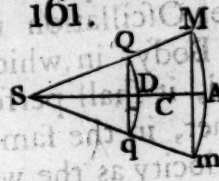
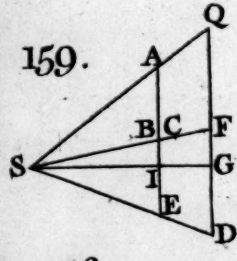
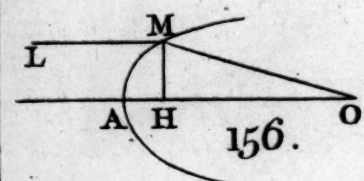
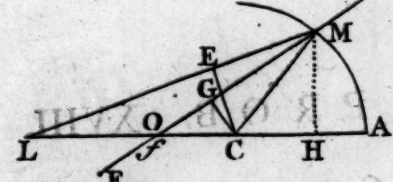
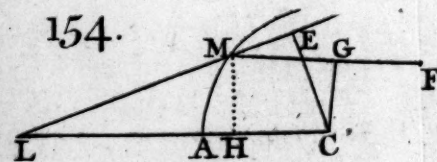
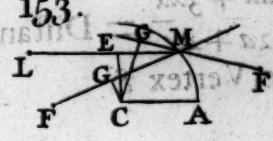
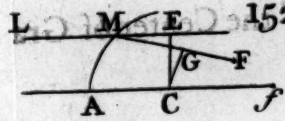
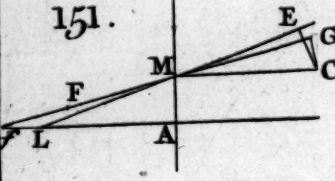
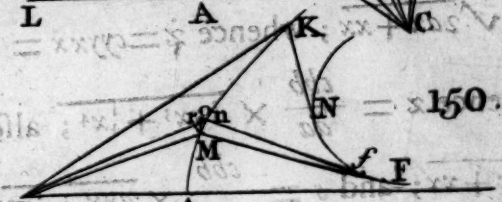
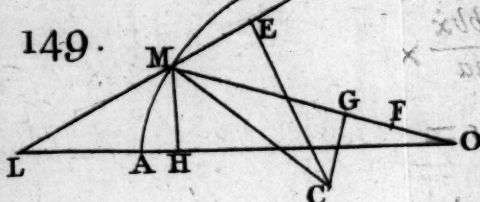
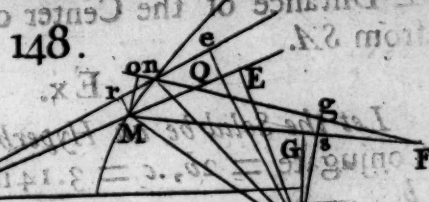
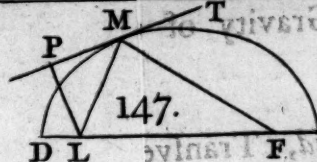
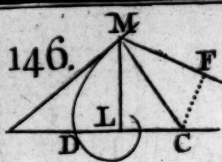
$\frac{\text{Sum of } Cn \times p}{\text{Sum of } Cp^2 \times p} = \frac{Cr}{CO^2}$: But by the Nature of the Center of Gravity, the Sum of all

the $Cn \times p = Cg \times \text{Body } AD = \frac{Cg}{CG} \times CG \times \text{Body } AD$

$= \frac{Cr}{CO} \times \text{Sum of all the } Cf \times p$. Therefore $\frac{Cr}{CO}$

$\times \frac{\text{Sum } Cf \times p}{\text{Sum } Cp^2 \times p} = \frac{Cr}{CO^2}$; whence $CO =$

Sum



Sum of all $Cp^2 \times p$: And therefore the Center of
 Sum of all $Cf \times p$: And therefore the Center of
 Oscillation is the same with the Center of Percussion.
 Since Sum of $Cf \times p = CG \times \text{Body } AD$; therefore

$$\frac{\text{Sum of } Cp^2 \times p}{CG \times \text{Body } AD} = CG$$

Also since $Cp^2 = CG^2 + Gp^2 - 2CG \times Gf$, therefore the
 Sum of all $Cp^2 \times p = \text{Sum of all } CG^2 + Gp^2 \times p - \text{Sum of all } 2CG \times Gf \times p$. But by the Nature of the Center
 of Gravity, the Sum of all the $Gf \times p = 0$; therefore
 $\text{Sum of } Cp^2 \times p = \text{Sum } CG^2 \times p + \text{Sum } Gp^2 \times p = CG^2 \times \text{Body } AD + \text{Sum } Gp^2 \times p$. Therefore $CO =$

$$\frac{\text{Sum of } Cp^2 \times p}{CG \times \text{Body } AD} = CG + \frac{\text{Sum of } Gp^2 \times p}{CG \times \text{Body } AD}$$
 And

$$= \frac{\text{Sum of all } Gp^2 \times p}{CG \times \text{Body } AD}$$

Therefore if $s = \text{Body}$, $Cp = x$, $Cf = v$, $Gp = z$,
 then the Sum of all the $Cp^2 \times p = \text{Sum of } x^2 s = \text{Fluent of } x^2 s$,
 and Sum of $Cf \times p = \text{Sum of } v s = \text{Fluent of } v s$

And Sum of $Gp^2 \times p = \text{Sum } z^2 s = \text{Fluent of } z^2 s$
 therefore $CO = \frac{\text{Fluent of } z^2 s}{\text{Fluent of } v s} = \frac{\text{Fluent of } z^2 s}{CG \times \text{Body } AD}$

$$CG + \frac{\text{Fluent of } z^2 s}{CG \times \text{Body } AD}$$
 Whence this

R U L E.

Draw a parallel Line to the Axis of Motion, through
 the Center of Gravity of the Figure (whether it be Line,
 Surface or Solid); and find the Sum of all the Products
 of every Particle of the Figure, multiply'd by the Square
 of its Distance from the Axis of Motion, or else from
 the parallel Line; which is done thus,
 1. Multiply the (Moment or) Fluxion of the Figure,
 by the Square of its Distance from the Axis of Motion,
 and

FIG. and (by Help of the Equation of the Curve) find the
 171. *Fluent F*; or else multiply by the Square of its Distance from the parallel Line, and find the *Fluent G*.

But in many Cases, especially of Solids, the *Fluent* cannot be had at one Operation; and then you must first find the Sum of these Products in the generating Line or Plane, of the Figure proposed; thus. Multiply the (Moment or) Fluxion of that generating Line or Plane by the Square of its Distance from the Axis of Motion, or else from the parallel Line; and find the *Fluent*. And then multiply this by the Fluxion of the Abscissa of the Figure; and find the *Fluent F*, or else *G*.

2. Then draw a Line thro' the Point of Suspension and the Center of Gravity; and from the generating Point Line, or Plane of the Figure, let fall a Perpendicular upon it; and find the Distance of that Perpendicular from the Point of Suspension; then multiply the Fluxion of the Figure by that Distance, and (from the Equation of the Figure) find the *Fluent M*. And let d = Distance from the Point of Suspension to the Center of Gravity, and B = Body or Figure given. Then $\frac{F}{M}$ or $\frac{F}{dB}$ or $+\frac{G}{dB}$ will be the Distance of the Center of Percussion or Oscillation from the Point of Suspension.

SCHOLIUM.

If the Center of Percussion or Oscillation be made the Center of Suspension, then the former Point or Center of Suspension, becomes the Center of Percussion; if the Plane of its Motion remains the same. And in general, the Distance of the Center of Suspension from the Center of Percussion or Oscillation, in the same Body, will always remain the same; if the Distance of its Center of Gravity from the Point of Suspension, and the Plane of its Motion (in regard to the Body) remain the same. For then d and

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FIG.

171.

G remain the same, and $\frac{G}{dB} =$ Distance of the Centers of Gravity and Percussion. And therefore in this Case if the Body be put into any oblique Position, it makes no Difference. Likewise if the Distances of the Center of Gravity from the Centers of Suspension and Oscillation, be known in one Case, they are known in all Cases, the Plane of the Motion remaining the same. For their Rectangle $= \frac{G}{B}$; and therefore they are reciprocally proportional. And hence, as to the Calculation, it will be sufficient to find the Center of Oscillation in the simplest Case, and then it may be easily had in any other, or when the Point of Suspension is at any other Distance assigned.

COR. The Center of Pressure of any Plane immersed in a Fluid, and sustaining that Fluid, is the same with the Center of Percussion of that Plane; the Axis of Motion being the Intersection of this Plane with the Surface of the Fluid. The Center of Pressure is that Point against which a Force being applied $=$ Sum of all the Pressures, shall just sustain them; so as the Plane shall incline to neither Side.

171.

Through the Center of Gravity of the Plane draw AO perpendicular to AS the Intersection of the Plane and Surface of the Fluid, and let cd be parallel to AS . Then the Pressure against any small Part cd is as $cd \times Ab$, and its Force to turn the Plane about the Center of Pressure is $cd \times Ab \times bO = cd \times Ab \times AO - cd \times Ab^2$, and the Sum of all these must be equal to 0, therefore $AO = \frac{\text{Sum of } cd \times Ab^2}{\text{Sum of } cd \times Ab}$, therefore O is the same with the Center of Oscillation and Percussion, and consequently is to be found the same Way.

172.

S f

EXAMPLE

EXAMPLE I

FIG. Let CB be a right Line, $CB = x$, then $\dot{x} = x^2 \dot{x}$
 173. and $F = \frac{x^3}{3}$; also $\dot{M} = x \dot{x}$, and $M = \frac{x^2}{2}$;
 whence $\frac{F}{M} = \frac{2}{3}x = CO$.

Ex. 2.

174. In a Parallelogram where the Axis of Motion is in the Plane of the Figure, $CB = x$, $BD = b$, then $\dot{x} = bx^2 \dot{x}$, and $F = \frac{bx^3}{3}$; also $\dot{M} = bx \dot{x}$, and $M = \frac{bx^2}{2}$; whence $\frac{F}{M} = \frac{2}{3}x = CO$.

Ex. 3.

175. Let ABD be the Arch of a Circle, the Center C the Point of Suspension, and the Axis of Motion perpendicular to its Plane; let Arch $ABD = s$, Cord $AD = c$, $CB = r$. Then $F = rrs$; and by Ex. 3. Prob. XVII. $d = \frac{rc}{s}$; therefore $\frac{F}{dB} = \frac{F}{ds} = \frac{rs}{c}$.

Ex. 4.

176. Let AD be a right Line, the Axis of Motion perpendicular to the Plane passing through it. $CB = d$, $BA = y$, then $\dot{x} = \overline{dd + yy} \times 2\dot{y}$, and $F = 2ddy + \frac{2}{3}y^3$; also $M = 2dy$; whence $\frac{F}{M} = \frac{2ddy + \frac{2}{3}y^3}{2dy} = d + \frac{yy}{3d}$.

Ex. 5.

177. For the Periphery of a Circle, let $CD = d$, Radius $AD = r$, Circumference $= c$. If the Axis of Motion be perpendicular to its Plane, then $G = rrc$, and $d + \frac{G}{dB} = d + \frac{rrc}{dc} = d + \frac{rr}{d}$.

But

But if the Axis of Motion be in the Plane of the F I G. Circle, let $DE = z$, $PQ = s$, then $\dot{G} = z^2 \dot{s} =$

$$\frac{4rz^2\dot{z}}{\sqrt{rr-zz}}, \text{ and by Form 17, the Fluent } G = \frac{rrc}{2}, \text{ for}$$

$$\text{the whole Circle: Therefore } d + \frac{G}{dB} = d + \frac{rr}{2d}.$$

Ex. 6.

For the Plane of the Circle, the Axis of Motion perpendicular to its Plane, $DE = z$, Circumference at E

177.

$$= \frac{cz}{r}, \text{ then } \dot{G} = \frac{cz^3\dot{z}}{r}, \text{ and } G = \frac{cz^4}{4r} = \frac{cr^3}{4}.$$

$$\text{Then } d + \frac{G}{dB} = d + \frac{cr^3}{4dB} = d + \frac{rr}{2d}.$$

And if the Axis of Motion be in the Plane of it,

$$\dot{G} = 4z^2\dot{z}\sqrt{rr-zz}, \text{ whence } G = \frac{cr^3}{8}; \text{ and } d +$$

$$\frac{G}{dB} = d + \frac{cr^3}{8dB} = d + \frac{rr}{4d}.$$

Ex. 7.

For the Periphery of the Circle, parallel to the Horizon, let the Axis of Motion be parallel to ED,

178.

Radius $DB = r$, $BA = v$, $DL = z$, $CD = d$, then $\dot{G} =$

$$4z^2\dot{z} = \frac{4rzz\dot{z}}{\sqrt{rr-zz}}, \text{ and (by Form 17) the whole}$$

$$\text{Fluent } G = \frac{rrc}{2}; \text{ then } d + \frac{G}{dB} = d + \frac{rrc}{2dc} =$$

$$d + \frac{rr}{2d}.$$

Ex. 8.

For the Plane of the Circle, whose Point of Suspension is in CD perpendicular to its Plane. Let AL

178.

$$= x, DL = z, \text{ then } \dot{G} = 4xz^2\dot{z} = 4z^2\dot{z} \times \sqrt{rr-zz},$$

S f 2

and

F I G.

and the whole Fluent $G = \frac{cr^3}{8}$; and $d + \frac{G}{dB} =$

$$d + \frac{rr}{4d}.$$

Ex. 9.

179.

In an Isosceles Triangle where the Axis of Motion is parallel to the Base, let $CD = a$, $CA = x$, $BE = f$, then

$$IK = \frac{fx}{a}, \text{ and } \dot{F} = \frac{fx^3 \dot{x}}{a}, \text{ whence } F = \frac{fx^4}{4a}; \text{ also}$$

$$\dot{M} = \frac{fx^2 \dot{x}}{a}, \text{ and } M = \frac{fx^3}{3a}; \text{ therefore } \frac{F}{M} = \frac{3}{4}x = \frac{3}{4}a.$$

If the Axis at C be perpendicular to the Plane of the Triangle, let $AQ = v$, then $CQ \times$ Fluxion of $AQ = xx + vv \times 2\dot{v}$, and x being given, the Fluent =

$$2x^2v + \frac{2v^3}{3}, \text{ then } \dot{F} = 2x^2v\dot{x} + \frac{2v^3\dot{x}}{3} = (\text{because}$$

$$v = \frac{fx}{2a}) \frac{fx^3\dot{x}}{a} + \frac{f^3x^3\dot{x}}{12a^3}; \text{ whence } F = \frac{fx^4}{4a} + \frac{f^3x^4}{48a^3} = \frac{12fa^3 + f^3a}{48}; \text{ therefore } \frac{F}{M} = \frac{12aa + ff}{16a}.$$

Ex. 10.

180.

In the Parabola CAF, $AC = x$, $AF = y$, $AQ = v$, $ax = yy$. Let the Axis in C be parallel to AF, then

$$\dot{F} = yx^2\dot{x} = x^2\dot{x}\sqrt{ax}, \text{ and } F = \frac{2}{3}x^3\sqrt{ax}; \text{ and } \dot{M} = yxx\dot{x} = xx\dot{x}\sqrt{ax}, \text{ and } M = \frac{2}{3}x^2\sqrt{ax}. \text{ Therefore}$$

$$\frac{F}{M} = \frac{3}{2}x.$$

If the Axis be perpendicular to its Plane, then $CQ \times AQ = xx + vv \times \dot{v}$, and the Fluent = $x^2v + \frac{1}{3}v^3$,

$$x \text{ being given; then } \dot{F} = x^2v\dot{x} + \frac{v^3\dot{x}}{3} = yx^2\dot{x} + \frac{y^3\dot{x}}{3} = x^2\dot{x}\sqrt{ax} + \frac{axx\dot{x}\sqrt{ax}}{3}; \text{ whence } F = \frac{2}{3}x^3\sqrt{ax} +$$

$\frac{1}{2}ax^2 \sqrt{ax}$. Therefore $\frac{F}{M} = \frac{1}{2}x + \frac{1}{2}a$.

Ex. 11.

Let AB be the Surface of a Sphere, $AB=s$, $BE=z$, 181.
Radius $AD=r$, c = Circumference. Then Circum-

ference of $BE = \frac{cz}{r}$, and $\dot{c} = \frac{cz^3 \dot{z}}{r} = \frac{cz^3}{r} \times$

$\frac{r\dot{z}}{\sqrt{rr-zz}}$; whence $G = \frac{2cr^3}{3} - \frac{2cr^2}{c} - \frac{czz}{3}$

$\sqrt{rr-zz} = \frac{2cr^3}{3}$. And $B=rc$; therefore $d + \frac{G}{dB}$

$= d + \frac{2rr}{3d}$.

Ex. 12.

Let $ABQD$ be a Parallelepipedon, the Axis of Mo- 182.
tion perpendicular to the Plane $ABQD$; let $AB=2a$,

$AD=2b$, Breadth = c , $GS=x$, $SZ=y$; then $GZ^2 \times$

$\dot{GZ} = xx+yy \times 4cy$, and the Fluent = $4cx^2y + \frac{4}{3}cy^3$,

being given; whence $G = 4cx^2yx + \frac{4cy^3x}{3} =$

$4cbx^2\dot{x} + \frac{4}{3}cb^3\dot{x}$; therefore $G = \frac{4}{3}cbx^3 + \frac{4}{3}cb^3x =$

$\frac{4}{3}cba^3 + \frac{4}{3}cb^3a$; also $B = 4acb$: Therefore $d +$

$\frac{G}{dB} = d + \frac{aa+bb}{3d}$.

Ex. 13.

In a Cylinder let $CA=x$, $AD=r$, $Ae=y$, $CH=a$, 183.
 c = Circumference, the Axis of Motion parallel to

AB ; then for the Circle ADB , $Ce^2 \times De \times \dot{Ae} =$

$xx+yy \times 4y\sqrt{rr-yy}$, and the whole Fluent = $\frac{rcx^2}{2}$

$+ \frac{r^3c}{8}$. Then $\dot{F} = \frac{rcx^2\dot{x}}{2} + \frac{r^3c\dot{x}}{8}$, and $F = \frac{rcx^3}{6}$

$+ \frac{r^3cx}{8}$; which corrected gives $F = \frac{rcx^3 - rca^3}{6}$

+

FIG.

$$+ \frac{r^3 cx - r^3 ca}{8}; \text{ and } B = x - a \times \frac{rc}{2}, d = \frac{x+a}{2}$$

$$\text{then will } \frac{F}{dB} = \frac{2x^3 - 2a^3}{3x^2 - 3aa} + \frac{rrx - rra}{2x^2 - 2aa}$$

$$\frac{4xx + 4ax + 4aa + 3rr}{6x + 6a}$$

Ex. 14.

184.

Let CKB be a Pyramid, whose Base is a Parallelogram, and Axis of Motion in C parallel to the Side D. Let its Altitude = a , $AB = f$, $AD = c$, $CH = x$, $HL =$

$$\text{then } KI = \frac{cx}{a}, \text{ and } EF = \frac{fx}{a}, \text{ and } GL \times KI$$

$$\frac{HL}{HL} = \frac{xx + yy}{xx + yy} \times \frac{cxy}{a}, \text{ and the Fluent} = \frac{cx^3y}{a}$$

$$\frac{cy^3x}{3a}; \text{ whence } \dot{F} = \frac{cx^3x}{a}y + \frac{cxy^3}{3a} = \frac{cfx^4x}{aa}$$

$$\frac{cf^3x^4x}{12a^4}, \text{ And } F = \frac{cfx^5}{5aa} + \frac{cf^3x^5}{60a^4}, \text{ Also } M$$

$$\frac{cfx^3x}{aa}, \text{ and } M = \frac{cfx^4}{4aa}; \text{ whence } \frac{F}{M} = \frac{fx}{3x}$$

$$\frac{ff}{15aa} = \frac{12aa + ff}{15a}$$

Ex. 15.

185.

In a right Cone, let $CA = g$, Altitude = a , Radius of the Base = f , $AB = x$, $BI = z$, $c = 3, 1416$ the

$$BE \text{ or } BD = \frac{fx}{a}, IE = \sqrt{\frac{ffxx}{aa} - zz}; \text{ then for}$$

$$\text{the Plane } DEB, CI \times IE \times 4BI = g + x^2 + zz \times x$$

$$\sqrt{\frac{ffxx}{aa} - zz}, \text{ and the Fluent} = \frac{g + x^2}{a} \times \frac{cffx^2}{aa}$$

$$+ \frac{cf^2x^4}{4a^4}, x \text{ being given. Then } \dot{F} = \frac{g + x^2}{a} \times \frac{cffx^2}{aa}$$

$$\frac{ffx^2x}{aa} + \frac{cf^4x^4x}{4a^4}, \text{ consequently } F = : \frac{1}{3}ggcx^3 + \frac{1}{4}cx^4 + \frac{cfff^5}{20aa} : \times \frac{ff}{an}. \text{ Also } M = \overline{g+x} \times \frac{ffx^2x}{aa}, \text{ and } M = : \frac{1}{3}gctx^3 + \frac{1}{4}cx^4 : \times \frac{ff}{aa}. \text{ Whence}$$

$$\frac{F}{M} = \frac{20gg + 30ga + 12aa + 3ff}{20g + 15a}.$$

Ex. 16.

For the Sphere AS. Draw the Diameter AS parallel to the Axis of Motion at C. And let AD = r, AE = v, Ef = z, EB = y, c = 3.1416: Then for the Circle BE, $c \times 2Ef \times \overline{EF} \times Ef^2 = 2cz^3z$; and the Fluent = $\frac{1}{2}cz^4 = \frac{1}{2}cy^4$; whence $\dot{G} = \frac{1}{2}cy^4\dot{v} = \frac{1}{2}c\dot{v} \times \overline{rv - vv^2} = 2cr^2v^2\dot{v} - 2crv^3\dot{v} + \frac{1}{2}cv^4\dot{v}$; and $G = \frac{2}{3}cr^3v^3 - \frac{1}{2}crv^4 + \frac{1}{10}cv^5$. Also the Fluxion of the Solid = $cyy\dot{v} = \dot{v} \times \overline{2rv - vv}$, and the Solid or the Segment BAE = $crv^2 - \frac{1}{3}cv^3 = B$. Then $\frac{G}{dB} = \frac{2cr^3v - 15rv^2 + 3v^3}{d \times 30r - 10v}$. And for the whole Sphere,

$$\frac{G}{dB} = \frac{2rr}{5d} = DO; \text{ and } CO = d + \frac{2rv}{5d}.$$

If v be very small, then $DO = \frac{2rv}{3d} = \frac{yy}{3d}$, nearly; and in most Clocks the Bob of the Pendulum is in this Form.

Ex. 17.

Let AD be a Paraboloid; CA = a, AB = x, BD = y, BI = z, c = 3.1416, rx = yy. Then $EI \times \overline{BI} \times 4CF = \overline{a+x^2} + zz : \times 4\dot{x}\sqrt{yy-zz}$, and the Fluent = $\overline{a+x^2} \times cy + \frac{1}{4}cy^4$; therefore $\dot{F} = \overline{a+x^2}^2 \times cyy\dot{x} + \frac{cy^4}{4}$.

F I G. $\dot{x} = \overline{a+x}^2 \times crx\dot{x} + \frac{1}{2}cr^2x^2\dot{x}$; whence $F = \frac{1}{2}ca\dot{x}$
 $+ \frac{3}{2}carx^3 + \frac{1}{2}crx^4 + \frac{1}{2}crrx^3$. Also $\dot{M} = \overline{a+x} \times c$
 $= \overline{a+x} \times crx\dot{x}$, and $M = \frac{1}{2}carx^2 + \frac{1}{2}crx^3$. Therefore

$$\frac{F}{M} = \frac{6a^2 + 8ax + rx + 3xx}{6a + 4x}$$
.

P R O B. XIX.

187. To find the Law of centripetal Force requisite to cause
 Body to move in a given Curve BF.

Let B be the Place of the Body moving in the
 Orbit BF by a Force directed to the given Point C .
 Draw the Tangent BY , and the Radii CB , CQ , infinitely
 near each other. QR parallel to CB ; and CY , Qn
 Perpendiculars to BY . Let the Distance CQ
 $= D$, perpendicular $CY = P$, then the infinitely small
 Line QR will be as the Force and Square of the Time
 conjunctly, that is as the Force and Square of the
 Area CBQ ; therefore the Force is as $\frac{QR}{CB^2}$ or $\frac{QR}{QB^2 \times P}$

that is (because $P : D :: Qn : QR = \frac{Qn \times D}{P}$) as

$\frac{Qn \times D}{QB^2 \times P^3}$. But $\frac{QB^2}{2Qn} =$ Radius of Curvature in

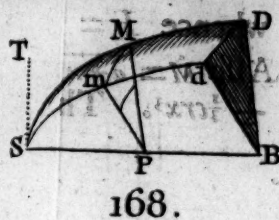
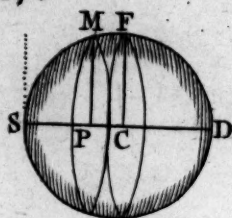
the Point B , and the same Radius is also $\frac{D\dot{D}}{\dot{P}}$ by

Prob. V. wherefore the Force is as $\frac{\dot{P}}{2P^3\dot{D}}$, that is

(supposing $\dot{D}$ to be given) as the Fluxion of $\frac{-1}{PP}$.

Therefore to find the Law of centripetal Force, let
 $D =$ Distance from the Center of Force, $P =$ perpendicular
 cular

167.

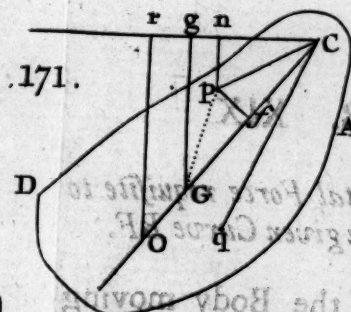
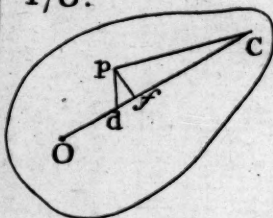


168.

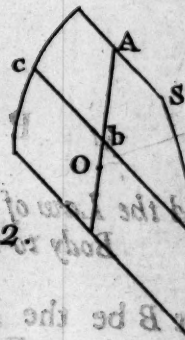


169.

170.

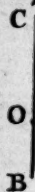


171.

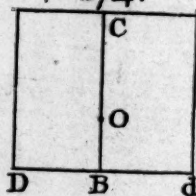


172.

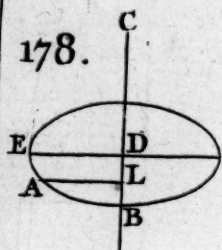
173.



174.



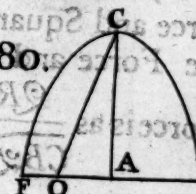
178.



179.



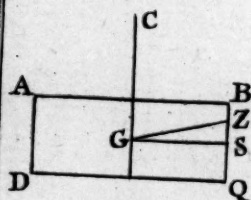
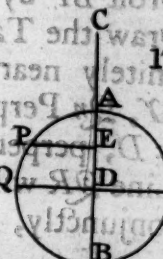
180.



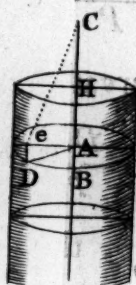
181.



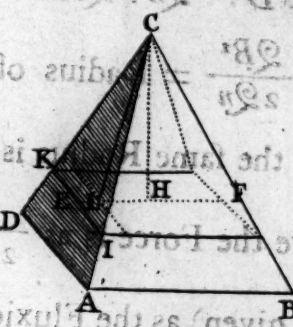
177.



182.



183.



184.



185.



186.

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lar on the Tangent: Compute the Value of P in FIG.
ms of D, by the Nature of the Curve; then find the
uxion of $\frac{-1}{PP}$, making $\dot{D} = 1$, and then expunge
Quantities as far as possible, except D, and you will
ve F, the Law of centripetal Force required.

EXAMPLE I.

Let C the Center of Force be in the Circumference 188.
 the Circle CBD; Diameter $CD = 2r$, the Triangles
 BD and CBY are similar, whence $P = \frac{DD}{2r}$, there-
 re $\frac{-1}{PP} = \frac{-4rr}{D^4}$, therefore $F \propto \frac{16r^2\dot{D}}{D^5} \propto \frac{1}{D^5}$;
 at is, the Force is reciprocally as the Fifth Power
 D.

Ex. 2.

Let DB be a Circle; and C at an infinite Distance, 189.
 $E = x$, $ED = y$, $AD = r$, then $P = \frac{Dy}{r}$, and $\frac{-1}{PP}$
 $\frac{-rr}{DD} y^{-2}$, whose Fluxion (because D is a standing
 quantity) is $\frac{2r^2}{DD} y^{-3} \dot{y}$, therefore $F \propto \frac{2r^2}{D^2} y^{-3}$ or
 $\propto \frac{1}{y^3}$.

Ex. 3.

Let BF be an Ellipsis; C the Focus, E the Center; 190.
 w BE and its Conjugate AE, and let the transverse
 is $= 2r$, Conjugate $= 2c$; then by the Property
 the Ellipsis $2rD - DD$ the Rectangle of the focal
 stances from B, is $= AE^2$, also AE or $\sqrt{2rD - DD}$
 $\therefore D : P = \frac{cD}{\sqrt{2rD - DD}}$; and $\frac{-1}{PP} = \frac{-2rD + DD}{ccDD}$
 T t =

F I G. $= \frac{-2r}{ccD} + \frac{1}{cc}$, whose Fluxion is $\frac{2r\dot{D}}{ccDD}$, therefore

$$F \propto \frac{2r}{ccDD} \text{ or } \frac{1}{DD}.$$

Ex. 4.

191. Let C be the Center of the Ellipsis; then by the Nature of the Figure $AC^2 + CB^2 = rr + cc$, and $AC = \sqrt{rr + cc - DD}$; also $AC : c :: r : P = \frac{cr}{\sqrt{rr + cc - DD}}$; therefore $-\frac{1}{PP} = \frac{-rr - cc + DD}{ccrr}$, whose Fluxion is $\frac{2D\dot{D}}{ccrr}$; therefore $F \propto \frac{2D}{ccrr} \propto D$. That is, the Force is as the Distance.

Ex. 5.

192. Let BA be an Hyperbola, C the Focus. Proceeding as in the Ellipsis, we shall find $P = \frac{cD}{\sqrt{2rD + DD}}$; and $-\frac{1}{P^2} = \frac{-2rD - DD}{ccDD} = -\frac{2r}{ccD} - \frac{1}{cc}$, whose Fluxion is $\frac{2r\dot{D}}{ccDD}$, therefore $F \propto \frac{2r}{ccDD}$ or $F \propto \frac{1}{DD}$.

And after the same Manner if the Force be in the other Focus, there will be found $P = \frac{cD}{\sqrt{DD - 2rD}}$,

and the Force $F \propto \frac{-2r}{ccDD} \propto \frac{-1}{DD}$, and is therefore a centrifugal Force,

Ex. 6.

193. Let AB be an Hyperbola, C the Center, Transverse $= 2r$, Conjugate $= 2c$, b = half the Conjugate belonging to CB . Then by the Property of the Hyperbola

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F I G.

$bb-DD=rr-cc$, and $b=\sqrt{rr-cc+DD}$; and

$b:c::r:P=\frac{cr}{\sqrt{rr-cc+DD}}$. Consequently —

$\frac{1}{PP}=\frac{-rr+cc-DD}{ccrr}$, whose Fluxion is $\frac{-2D\dot{D}}{ccrr}$,

therefore $F\propto\frac{-2D}{ccrr}\propto-D$, and is therefore a centrifugal Force.

Ex. 7:

194.

Let AB be a Parabola, C the Focus, r = Latus Rectum; by the Property of the Figure $P=\sqrt{\frac{rD}{4}}$,

whence — $\frac{1}{PP}=\frac{-4}{rD}$, whose Fluxion is $\frac{4\dot{D}}{rDD}$;

therefore $F\propto\frac{1}{DD}$.

Ex. 8.

195.

Let AB be a Parabola, C the Center of Force at an infinite Distance in the Axis. $AD=x$, $DB=y$,

$ax=yy$, then $P=\frac{Dy}{\sqrt{4xx+ax}}$, and $\frac{-1}{PP}=\frac{-4x}{aDD}-\frac{1}{DD}$, whose Fluxion is

$\frac{-4\dot{x}}{aDD}+\frac{8x\dot{D}}{aD^3}+\frac{\dot{D}}{D^3}=(\text{because }-\dot{x}=\dot{D}):$

$\frac{4D}{aD^3}+\frac{8x+a}{aD^3}:\times\dot{D}$; therefore $F\propto\frac{4D+8x+a}{aD^3}$,

or (because D is infinite) $\propto\frac{4D}{aD^3}$, or $F\propto\frac{4}{aD^2}$ a given Quantity.

Ex. 9.

196.

Let C be the Vertex of the Parabola, $CA=x$, $AB=y$, $ax=yy$; by similar Triangles $TB:AB::TC$ or $x:$

Cr or $P=\frac{xy}{\sqrt{4xx+yy}}$; whence — $\frac{1}{PP}=-\frac{1}{4xx+yy}$

FIG. $\frac{4xx+yy}{xxyy} = \frac{-4}{ax} - \frac{1}{xx}$; whose Fluxion is $\frac{4\dot{x}}{ax^2} + \frac{2\dot{x}}{x^3}$. But $DD=xx+ax$, and thence $\dot{x} = \frac{2D\dot{D}}{a+2x}$, therefore $F \propto \frac{4x+2a}{ax^3} \times \frac{2D\dot{D}}{2x+a} = \frac{4D\dot{D}}{ax^3}$, or $F \propto \frac{D}{x^3}$.

EX. 10.

197. Let *VAR* be an Ellipsis, *C* the Focus; and let the Curve *VBQ* be formed from the Ellipsis, thus; take *CA=Cb*, and the Angle *VCA* to the Angle *VCb*, as *m* to *n*, for all the Points of the Curve. To find the Law of centripetal Force of a Body moving in the Curve *VBQ*.

Let the Transverse of the Ellipsis = *2r*, Conjugate = *2c*. By the Property of the Ellipsis the Perpendicular *Cp* = $\frac{cD}{\sqrt{2rD-DD}}$, also *tb*=*TA*, and $\angle BCb$

$$= \frac{n}{m} \angle ACo. \text{ And by similar Triangles } \frac{P \times tb}{BY} =$$

$$tB = \frac{n}{m} TO = \frac{\frac{n}{m} Cp \times TA}{pA}, \text{ that is } \frac{P}{\sqrt{DD-PP}}$$

$$= \frac{nc}{m\sqrt{2rD-DD-cc}}; \text{ whence by Reduction}$$

$$PP = \frac{n^2c^2D^2}{2rm^2D-m^2D^2-m^2c^2+n^2c^2}; \text{ whence } \frac{-1}{PP}$$

$$= \frac{-2rm^2}{n^2c^2D} + \frac{m^2}{n^2c^2} - \frac{n^2c^2-m^2c^2}{n^2c^2D^2}; \text{ whose Fluxion}$$

$$\text{is } \frac{2rm^2\dot{D}}{n^2c^2D^2} + \frac{2n^2-2m^2}{n^2c^2D^3}c^2\dot{D}, \text{ therefore } F \propto \frac{m^2}{DD} +$$

$$\frac{n^2-m^2}{rD^3}cc. \text{ In the same Manner if } C \text{ were the Center}$$

of the Ellipsis, it might be proved that the Force is

$$\text{as } \frac{m^2D}{rr} + \frac{nn-mm}{D^3}cc.$$

EX. 11.

EX. 11.

Suppose AB to be the Logarithmic Spiral; since the Angle CBT is always given, there is given the Ratio FIG. 198.

of CB to CT suppose as m to n . Then $P = \frac{n}{m} D$;

and $\frac{-I}{PP} = \frac{-mm}{nnDD}$, whose Fluxion is $\frac{2m^2\dot{D}}{nnD^3}$;

therefore $F \propto \frac{1}{D^3}$.

EX. 12.

Let CB be the hyperbolic Spiral; draw CT perpendicular to CB , and let $CB = y$, the given Subtangent 199.

$CT = a$. By similar Triangles $\sqrt{aa + yy} : y :: a : P$

$= \frac{ay}{\sqrt{aa + yy}}$; and $-\frac{I}{PP} = -\frac{I}{yy} - \frac{I}{aa}$,

whose Fluxion is $\frac{2\dot{y}}{y^3} = \frac{2\dot{D}}{D^3}$, therefore $F \propto \frac{1}{y^3}$

$\propto \frac{1}{D^3}$.

PROB. XX.

The Nature of the Curve ABD forming an Arch being given; to find the Nature of the Curve RST bounding the Top of the Wall $ATRD$ supported by that Arch; by the Pressure or Weight of which Wall, all the Parts of the Arch are kept in Equilibrio without falling.

200.

1. Let several equal right Lines $AB, BC, CD, \&c.$ placed in a vertical Plane, be moveable round the Angles $A, B, C, D, \&c.$ whilst the Points A, G , at the Base remain fixed and immoveable. Through B ,

201.

~~210.~~
201. FIG. *B, C, D, &c.* draw the Lines *Bi, Cm, Dp, &c.* perpendicular to the Horizon; and complete the Parallelogram *Bbik*, and make $Cl=Bk$, and complete the Parallelogram *Clmn*. In like Manner make $Do=Cn$ or lm , $Er=op$, $Ft=rs$, and complete all the Parallelograms in the Figure as at first.

2. Let several Weights which are to one another as the Lines *Bi, Cm, Dp, &c.* lie respectively on the Points *B, C, D, &c.* Now the Force *Bi*, is equivalent to *Bb, Bk*, acting in the Directions *BA, BC*; the Force *Bb* is destroyed by the Resistance of the Point *A*; but *Bk* endeavours to move the Point *B* towards *C*. In like Manner the Force *Cm* is equivalent to *Cl* and *Cn*; the Force *Dp* to *Do, op, &c.* Now the Forces *Bk* (acting towards *C*) and *Cl* (acting towards *B*) being equal by Construction destroy one another. In like Manner the Forces *Cn, Do, Dq*, and *Er; Ev* and *Ft, &c.* destroy one another; and the Point *G* being fixed, it is manifest the Figure *ABCD, &c.* will not be moved by the incumbent Weights, *Bi, Cm, Dp, &c.* but all its Parts will remain in Equilibrio.

3. The Force *Bb* : Force *Bk* or *Cl* :: Sine *L bi B* or *i BC* : *S. L ABi* :: $\frac{1}{S. ABi} : \frac{1}{S. iBC}$ or $\frac{1}{S. mCB}$

Likewise Force *Cl* : Force *Cn* or *Do* :: *S. mCD* or

pDC : *S. mCB* :: $\frac{1}{S. mCB} : \frac{1}{S. mCD}$ or $\frac{1}{S. pDC}$

and so on; whence it is plain in general, that any

Force *Cl* is as $\frac{1}{S. L. mCB}$. Now since *Cm* =

$\frac{S. Clm \times Cl}{S. Cml} = \frac{S. BCx \times Cl}{S. mCD}$; therefore the Force

$Cm \propto \frac{S. BCx}{S. mCB \times S. mCD}$.

4. Now let the Number of the Lines *AB, BC, CD, &c.* be increased and their Lengths diminished *ad infinitum*, that the Figure may obtain the Form of a Curve,

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F I G.

a Curve, and the Pressure will then act on all Parts of it; and the Angle BCx will then become the Angle of Contact, and the Sines of mCB and mCD become equal to the Sine of mCx : Therefore drawing the Tangent An (Fig. 200) the Pressure on any Point A to preserve the Equilibrium will be as the Angle of Contact at A directly, and the Square of the Sine of the Angle mAn reciprocally. But the Angle of Contact is as the Curvature, or reciprocally as the Radius of Curvature. Therefore the Pressure is reciprocally as that Radius and the Square of the Sine of that Angle mAn .

200.

5. Let $BC=x$, $AC=y$, $AB=z$. Radius of Curvature in $A=R$. Then if $\dot{z}$ be given, $S. L \text{ TAN} \propto$

$\dot{y}$, and $\frac{1}{S.TAN} \propto \frac{1}{\dot{y}}$. Then the Weight or Pres-

sure on $A=AT \times \dot{y}$, and that (as has been proved)

is as $\frac{1}{R \times S.TAN^2} \propto \frac{1}{R\dot{y}^2}$, therefore $AT \propto \frac{1}{R\dot{y}^3}$;

or in general $AT \propto \frac{\dot{z}^3}{R\dot{y}^3}$. But when $\dot{x}$ is given

$R = \frac{\dot{z}^3}{-\dot{x}\dot{y}}$; or if $\dot{y}$ be given $R = \frac{\dot{z}^3}{y\ddot{x}}$; there-

fore $AT \propto \frac{-\dot{x}\dot{y}}{\dot{y}^3}$ if $\dot{x}$ be given; or $AT \propto \frac{\ddot{x}}{\dot{y}^2}$, if $\dot{y}$ be given.

Wherefore to find the Curve ST , let $BC=x$, $AC=y$, $AB=z$; then by the Nature of the Curve AB , com-

pute $\frac{-\dot{x}\dot{y}}{\dot{y}^3}$ if $\dot{x}$ be given, or $\frac{\ddot{x}}{\dot{y}^2}$ if $\dot{y}$ be given,

and take AT proportional thereto. And x may be expunged out of the Value of AT , by Help of the given Line BS , and thence the Nature of the Curve ST will be known.

EXAMPLE

EXAMPLE I.

FIG. Let BA be a Circle, Radius $AR=r$, $BC=x$, $AC=y$

202.

$BS=a$, $\dot{x}$ given. Then $y=\sqrt{2rx-xx}$, $\dot{y}=\frac{r-x}{\sqrt{2rx-xx}}\dot{x}$

$$\ddot{y} = \frac{-r\dot{x}^2}{(2rx-xx)^{\frac{3}{2}}}; \text{ then will } \frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{r\dot{x}^3}{(2rx-xx)^{\frac{3}{2}}}$$

$$\times \frac{(2rx-xx)^{\frac{3}{2}}}{r-x^3 \times \dot{x}^3} = \frac{rr}{r-x^3}; \text{ therefore } AT \propto \frac{rr}{r-x^3}$$

$$\text{or } AT = \frac{r^3 a}{r-x^3}. \text{ And expunging } x, S\mathcal{Q} =$$

$$(AT-CS) = \frac{r^3 a}{(rr-yy)^{\frac{3}{2}}} - a - r + \sqrt{rr-yy}, \text{ for}$$

the Nature of the Curve ST .

Hence the Curve ST runs upwards *ad infinitum*, and the Perpendicular DE is an Affymptote to the Curve.

SCHOLIUM.

If ST had been a right Line, then the Point A would be pressed with too little Weight, and B with too much: And hence appears the Reason why circular Arches commonly break about the Top, by being loaded there with more Weight than their due Proportion.

Ex. 2.

202.

Let the Curve DAB be an Ellipsis; $BR=r$, $DR=c$,

$CB=x$, $AC=y$, $BS=a$, $\dot{x}$ given. Then $y=\frac{c}{r}\sqrt{2rx-xx}$

$$\dot{y} = \frac{c\dot{x}}{r} \times \frac{r-x}{\sqrt{2rx-xx}}, \ddot{y} = \frac{-c\dot{x}^2}{(2rx-xx)^{\frac{3}{2}}}: \text{ Whence}$$

$$\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{r^4}{cc \times r-x^3}, \text{ which is as } AT; \text{ therefore}$$

$$AT = \frac{r^3 a}{r-x^3}. \text{ Then } S\mathcal{Q} = \frac{r^3 a}{r-x^3} - a - x \quad \text{F I G.}$$

$$= (\text{expunging } x) \frac{c^3 a}{cc-yy}^{\frac{3}{2}} - a - r + \frac{r}{c} \sqrt{cc-yy},$$

for the Nature of the Curve ST , being of the same Kind with the foregoing.

Ex. 3.

Let AB be a Parabola, $BC=x$, $CA=y$, $SB=a$, 203.

$$rx=yy, \dot{y} \text{ given. Then } \dot{x} = \frac{2y\dot{y}}{r}, \ddot{x} = \frac{2\dot{y}^2}{r}, \text{ and}$$

$\frac{\ddot{x}}{\dot{y}^2} = \frac{2}{r}$, a given Quantity, therefore AT is every where the same, and ST is the same Parabola placed in a higher Position.

Ex. 4.

Let BA be an Hyperbola, Transverse $= 2r$, Conjugate $= 2c$, $BC=x$, $CA=y$, $BS=a$, $\dot{y}$ given. Then 203.

$$2rx+xx = \frac{rryy}{cc}, \text{ whence } r+x = \frac{r}{c} \sqrt{cc+yy}, \dot{x} =$$

$$\frac{ry}{c} \times \frac{y}{\sqrt{cc+yy}}, \ddot{x} = \frac{rc\dot{y}^2}{cc+yy}^{\frac{3}{2}}; \text{ wherefore } AT \propto$$

$$\left(\frac{\ddot{x}}{\dot{y}^2}\right) = \frac{rc}{cc+yy}^{\frac{3}{2}}, \text{ and } AT = \frac{c^3 a}{cc+yy}^{\frac{3}{2}}.$$

Hence the Curve ST continually approaches nearer and nearer the Hyperbola: And $S\mathcal{Q} = (a+x-AT=)$

$$a-r + \frac{r}{c} \sqrt{cc+yy} - \frac{c^3 a}{cc+yy}^{\frac{3}{2}}, \text{ expresses the Na-}$$

ture of the Curve ST .

Ex. 5.

Let BA be a Cycloid, $BC=x$, $CA=y$, $BD=s$, 204.
 $BV=a$, $\dot{x}$ given. By the Property of the Curve $y=$

U u

s +

FIG.

$$s + \sqrt{ax - xx}, \text{ but } \dot{s} = \frac{a\dot{x}}{2\sqrt{ax - xx}}, \text{ then } \dot{y} = \frac{2a\dot{x} - 2x\dot{x}}{2\sqrt{ax - xx}} = \dot{x} \sqrt{\frac{a-x}{x}}, \text{ and } \ddot{y} = \frac{-a\dot{x}^2}{2x\sqrt{ax - xx}}$$

$$\text{therefore } \frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{r}{2 \times a - x} : \text{ Therefore } AT \propto \frac{1}{a-x} \propto \frac{1}{CV^2}.$$

Ex. 6.

203. Let BA be the Catenary, $BC=x$, $CA=y$, $BA=a$

$$BS=a, \dot{x} \text{ given; then } z = \sqrt{2rx + xx}, \dot{z} = \frac{r+x}{\sqrt{2rx + xx}}$$

$$\text{therefore } \dot{y} = (\sqrt{\dot{z}^2 - \dot{x}^2}) = \frac{r\dot{x}}{\sqrt{2rx + xx}}, \text{ and } \ddot{y} = \frac{-r\dot{x}^2 \times r + x}{(2rx + xx)^{\frac{3}{2}}}; \text{ whence } \frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{r+x}{rr}, \text{ therefore}$$

$$AT \text{ is as } r+x, \text{ or } AT = \frac{a}{r} \times r+x. \text{ Hence,}$$

1. If $a=r$, then $AT=a+x=CS$, and the ST is a right Line passing through S .

2. If BS is very small, draw At perpendicular to the Curve, and by similar Triangles $(\dot{z}) = \frac{r+x \times \dot{x}}{\sqrt{2rx + xx}}$

$$(\dot{y}) = \frac{r\dot{x}}{\sqrt{2rx + xx}} : : (AT) = \frac{a}{r} \times r+x : At$$

$a=BS$; therefore the Arch is of the same Thickness every where: Consequently a heavy flexible Line put into this Figure would support itself.

3. For the Nature of the Curve ST , we have $S = (a+x-AT) = x - \frac{ax}{r}$. Therefore if r is less than r the Curve is concave towards B , and if a is greater than r , it is convex towards B .

Ex.

Ex. 7.

Let AB be the logarithmic Curve, GD its Af- F I G.
 symptote, $BS = a$, $BD = r$, Subtangent $GE = t$, 205.
 $BC = x$, $CA = y$, $AG = r + x$, then by the Property

of the Curve $\dot{y} = \frac{t\dot{x}}{r+x}$, and if $\dot{x}$ be given, $\ddot{y} =$
 $\frac{-t\dot{x}^2}{r+x^2}$, whence $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{t\dot{x}^3 \times \overline{r+x^3}}{r+x^2 \times t^3 \dot{x}^3} = \frac{r+x}{tt}$,

whence $AT \propto \overline{r+x}$, or $AT = \frac{a}{r} \times \overline{r+x}$, the same as
 in the last Example.

For the Nature of the Curve, $SQ = (a + x - AT$
 $=) x - \frac{ax}{r}$, therefore if $a = r$, then ST becomes
 the Affymptote DG , and if a be less than r , the Curve
 is concave; but if a is greater than r , it is con-
 vex towards B .

Or thus for the Nature of the Curve, $GT = (r + x$
 $- AT =) r - a + \frac{r-a}{r}x$, for the Relation of
 ST to the Affymptote GD .

Ex. 8.

Let AB be the Cissoid, whose Equation is $axx - yxx$ 206.
 $= y^3$, in Fluxions $2ax\dot{x} - 2yxx\dot{x} - x^2\dot{y} = 3y^2\dot{y}$; this

again in Fluxions (making $\ddot{x} = 0$), $2a\dot{x}^2 - 2y\dot{x}\dot{x}^2 -$
 $xx\dot{y} - 2x\dot{x}\dot{y} - x^2\ddot{y} = 6y\dot{y}^2 + 3y^2\ddot{y}$; from the former
 Equation $\dot{y} = \frac{2ax - 2yx}{3yy + xx}\dot{x}$, and from the latter,

$$= \frac{2a - 2y \times \dot{x}^2 - 4x\dot{x}\dot{y} - 6y\dot{y}^2}{3yy + xx}.$$

Now if $a = 10$,

$= 1$; suppose $y = 2$, $x = 1$, to find AT ; here $\dot{y} =$
 $\dot{y} = -.5462$; whence $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = .293$ for the

U u 2

Value

F I G. Value of AT . And to find it in the Vertex where x and y are 0, and y infinitely greater than x ; we shall have $axx = y^3$, $\dot{y} = \frac{2ax}{3yy} = \frac{2}{3} \sqrt{\frac{a}{y}}$, $\ddot{y} = \frac{2a - 4x\dot{y} - 6y\dot{y}^2}{3yy}$
 $= \frac{-2a}{9yy}$; whence $\frac{-x\ddot{y}}{\dot{y}^3} = \frac{3}{4\sqrt{ay}} = \text{Infinity.}$

More generally thus :

Since $axx - yxx = y^3$, in Fluxions $2ax\dot{x} - 2yxx\dot{x} - x^2\dot{y} = 3y^2\dot{y}$, whence $\dot{x} = \frac{3xy^2 + x^3}{2y^3}\dot{y}$, this again in Fluxions and reduced (making $\dot{y}$ invariable), $\ddot{x} = \frac{3}{2} \times \frac{y^3 + x^2y \times \dot{x} - xy^2 - x^3 \times \dot{y}}{y^4} \times \dot{y} = (\text{expunging } xx)$
 $\frac{3}{2}\dot{y} \times \frac{ay\dot{x} - ax\dot{y}}{yy \times a - y} = (\text{expunging } \dot{x}) \frac{3aax\dot{y}^2}{4yy \times a - y^2} =$
 $(\text{expunging } x) \frac{3aay^2}{a - y^2 \times 4\sqrt{ay - yy}}$; whence $\frac{\ddot{x}}{\dot{y}^2} =$
 $\frac{3aa}{a - y^2 \times 4\sqrt{ay - yy}}$: Therefore $AT \propto \frac{1}{\sqrt{y} \times a - y^{\frac{3}{2}}}$.

P R O B. XXI.

207. *The Curve BA being given, by whose Revolution about the Axis BC there is generated a concave Surface of Vault; To find the Height AT of a Wall standing on the same, and supported by that Surface, so that all the Parts may remain in Equilibrio.*

Let PBQ be an infinitely small Part of the Surface contained between the Planes PBR and QBR , draw the Ordinates DC , AC ; and take Dd , Aa infinitely small

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small equal Parts of the Curve. Let $BC=x$, $CA=y$, $BA=z$: By the Reasoning in the last Problem, the Weight insitfing on the small Part of the Surface

$ADda$ will be as $\frac{1}{Ry^2}$, when the Particle of the

Curve is given. But this incumbent Weight is $ab \times AD \times AT$, But $AD \propto y$, therefore the Weight is as

$AT \times y$ and this $\propto \frac{1}{Ry^2}$, whence $AT \propto \frac{1}{Ry^3}$.

Or in general $AT \propto \frac{z^3}{Ryy^3}$. Whence

Putting $BC=x$, $CA=y$, $BA=z$. Take $AT \propto \frac{-\dot{x}\ddot{y}}{yy^3}$ if $\dot{x}$ be given, or $AT \propto \frac{\ddot{x}}{yy^2}$ if $\dot{y}$ is given:

And the Nature of the Curve ST will be known by expunging x .

EXAMPLE I.

Let BA be a cubic Parabola, $r^2x = y^3$; then $r^2\dot{x} = 3y^2\dot{y}$, and (if $\dot{y}$ be given) $r^2\ddot{x} = 6y\ddot{y}$, whence $\frac{\ddot{x}}{yy^2} = \frac{6}{rr}$; therefore AT is as $\frac{6}{rr}$ a given Quantity: Consequently the Curve ST is the same Parabola with BA , but placed in a higher Position. 203.

Ex. 2.

Let BA be a biquadratic Parabola, $r^3x = y^4$, and $\dot{y}$ given. Then $r^3\dot{x} = 4y^3\dot{y}$, and $r^3\ddot{x} = 12y^2\ddot{y}$, wherefore $\frac{\ddot{x}}{yy^2} = \frac{12y}{r^3}$, whence $AT \propto y$ or AC . 208.

Ex. 3.

Let BA be a Circle, $y = \sqrt{2rx - xx}$, $\dot{x}$ given; then $\dot{y} = \frac{rx - x\dot{x}}{\sqrt{2rx - xx}}$, $\ddot{y} = \frac{-rr\dot{x}^2}{(2rx - xx)^{\frac{3}{2}}}$; whence $-\dot{x}\ddot{y}$ 209.

F I G. $\frac{-\dot{x}\dot{y}}{\dot{y}^3} = \frac{rr}{y \times r - x^3}$; therefore $AT \propto \frac{1}{y \times r - x^3}$ the

is as $\frac{1}{AC \times CR^3}$. Whence the Perpendiculars RD and DE are Assymptotes to the Curve ST .

P R O B. XXII.

To find the Resistance of a plane Figure or Solid moving in a Fluid, in the Direction of its Axis.

210.

Let ABQ be any plane Figure or Solid whose Axis is AQ ; draw CBf parallel to the Axis AQ , and g and Ordinate BE Perpendiculars thereto; BD Tangent at B , and Df perpendicular to it. Call AB

x ; EB , y ; AB , z ; Bn , $\dot{z}$; Br , $\dot{x}$; rn , $\dot{y}$.

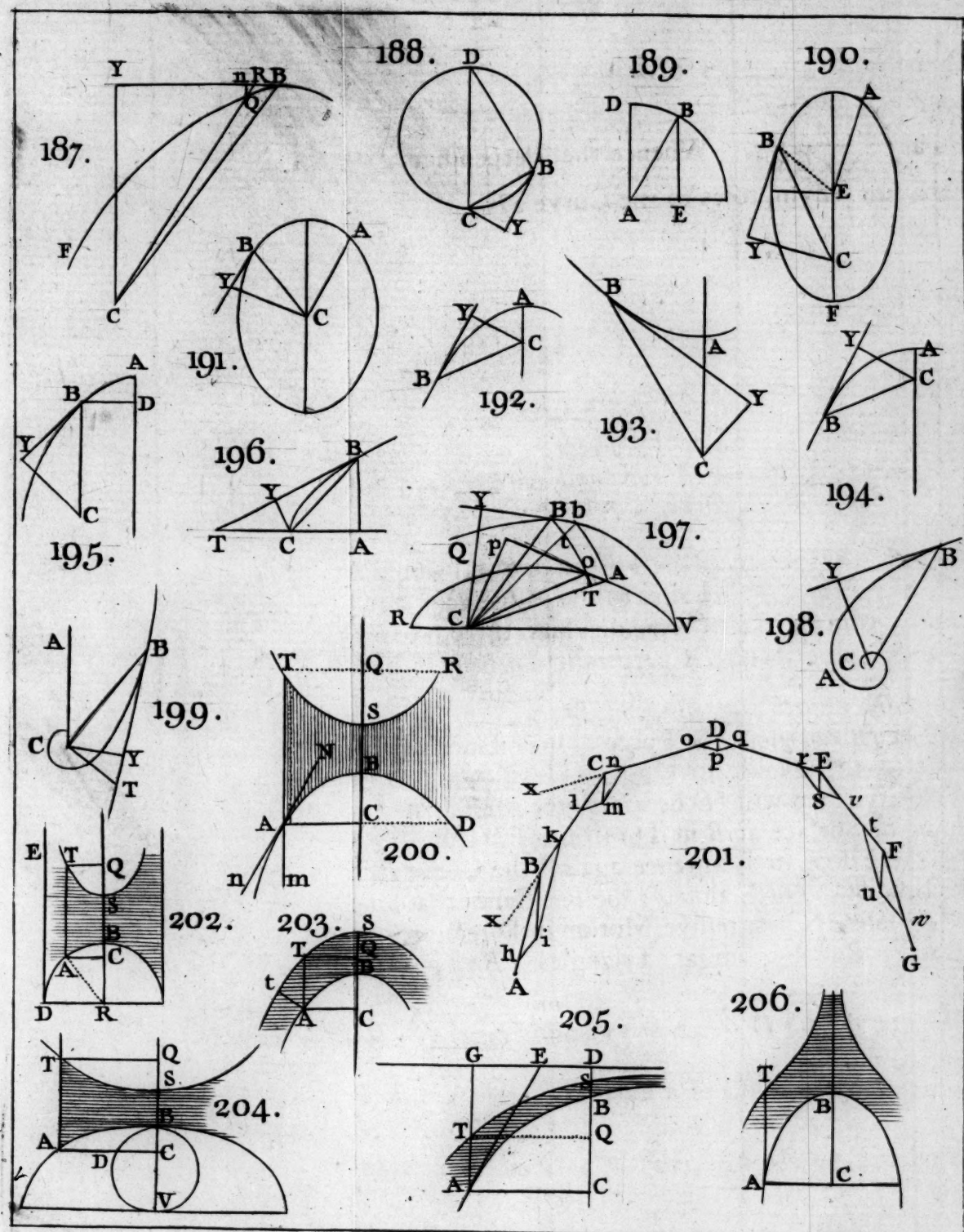
Let fB represent the Force or Resistance of a Particle of the Fluid, striking against C with a given Velocity, then will fD be the Force against the Curve Line or Surface at B in Direction fD ; and fg will be the Force or Resistance against the Curve in Direction BC , which alone is the Resistance that hinders or opposes its progressive Motion in Direction of the Axis. But by similar Triangles $fB : fD :: fB^2$

$$fD^2 :: DB^2 : Dg^2 :: \dot{z}^2 : \dot{y}^2, \text{ and } fg = \frac{\dot{y}^2}{\dot{z}^2} \times fB$$

Therefore the Force of a Particle against C and B are

as fB and $\frac{\dot{y}^2}{\dot{z}^2} \times fB$, that is as 1 to $\frac{\dot{y}^2}{\dot{z}^2}$. Now the

Quantity



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Quantity of Fluid striking against Bn in the Curve is $F I G$.

$\dot{y}$, and against Bn in the Solid (generated by AB re-

210.

volving round its Axis) as yy' : Therefore the Force

against the Base : Force against the Curve : : $\dot{y}$ to

$\dot{y}$, or as $\dot{y}$ to $\frac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$, or as $\dot{y}$ to $\frac{\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}}$, that

as $\dot{y}$ to $\frac{\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}}$: And Force against the Base :

Force against the Solid, is as yy' to $\frac{yy}{1 + \frac{\dot{x}^2}{\dot{y}^2}}$.

Note, by the Resistance of a plane Figure moving in a Fluid is meant the Resistance of a prismatic Solid of any given Depth, and whose Base is that Figure: and it is supposed to move in a Direction parallel to that Base.

Hence to find the Resistance; by the Equation of the

Curve, exterminate $\dot{x}^2$ out of the Quantity $\frac{\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}}$

for the Curve; and find the Fluent F ; or out of the

Quantity $\frac{yy}{1 + \frac{\dot{x}^2}{\dot{y}^2}}$ for the Solid, and find its Fluent

Then will the Resistance against the Base : to the Resistance against the Curve : : $y : F$. Or the Resistance against the Base : to the Resistance against the Solid : : as $y : 2G$.

EXAMPLE I.

Let there be a Triangle (or prismatic Solid) ARS moving in Direction QA . Let $AQ=b$, $QR=c$, $AR=d$, AE

211.

$AE=x$, $EB=y$; and by similar Triangles $x = \frac{by}{c}$,
whence $\dot{x}^2 = \frac{bb\dot{y}^2}{cc}$, therefore $\frac{\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}} = \frac{\dot{y}}{1 + \frac{bb}{cc}}$
 $= \frac{cc\dot{y}}{cc+bb} = \frac{cc\dot{y}}{dd}$, whose Fluent $F = \frac{ccy}{dd}$.
Whence the Resistance of the Base to that of the Side,
is as y to $\frac{ccy}{dd}$, that is as dd to cc .

Ex. 2.

211. Let ARS be a Cone, whose Axis is AQ ; then $x = \frac{by}{c}$, $\dot{x}^2 = \frac{bb\dot{y}^2}{cc}$, and $\frac{y\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}} = \frac{y\dot{y}}{1 + \frac{bb}{cc}} = \frac{ccy\dot{y}}{cc+bb} = \frac{ccy\dot{y}}{dd}$, whose Fluent $G = \frac{ccyy}{2dd}$, and
 $2G = \frac{ccyy}{dd}$. Therefore the Resistance of the Base
to that of the Side is as yy to $\frac{ccyy}{dd}$, or as dd to cc .

Ex. 3.

210. Let ABR be a Circle, Radius $AQ = r$, $AE = x$,
 $EB = y$. Then $\dot{x} = \frac{y\dot{y}}{r-x} = \frac{y\dot{y}}{\sqrt{rr-yy}}$, and $\frac{\dot{x}^2}{\dot{y}^2} = \frac{yy}{rr-yy}$;
therefore $\frac{\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}} = \frac{\dot{y}}{1 + \frac{yy}{rr-yy}} = \frac{rr-yy}{rr}\dot{y}$, whose Fluent $F = y - \frac{y^3}{3rr}$. Whence
the Resistance against the Base, to the Resistance
against the Circle (or cylindric Surface) is as y to
 $y - \frac{y^3}{3rr}$, that is as $3rr$ to $3rr-yy$: which when $y=r$
is as 3 to 2.

Ex. 4

Ex. 4.

Let ABQ be an Hemisphere, then $\frac{yy}{1 + \frac{\dot{x}^2}{y^2}} = \text{FIG. 210.}$

$\frac{rr - yy}{rr} \times yy$, whose Fluent $G = \frac{y^2}{2} - \frac{y^4}{4rr}$;

therefore the Resistance against the Base, to the Resistance against the convex Surface, is as y^2 to $y^2 - \frac{y^4}{2rr}$, or as $2rr$ to $2rr - yy$; which when $y = r$, is as 2 to 1.

Ex. 5.

Let $ABRQ$ be a Spheroid, Q the Center, $QA = a$, Latus Rectum $= 2r$, $AE = x$, $BE = y$, $QE = u = a - x$; 210.

then $uu = aa - \frac{a}{r}yy$, and $\dot{x} = -\dot{u} = \frac{ay\dot{y}}{r\sqrt{aa - \frac{a}{r}yy}}$;

therefore $\frac{yy}{1 + \frac{\dot{x}^2}{y^2}} = \frac{rra - ryy}{rra + a - r} \times yy$; whose

Fluent, by Form the 4th and 11th, is $G = \frac{-ryy}{2a - 2r}$

$+ \frac{\frac{1}{2}rraa \times 2.30258}{a - r^2} \times \text{Log} : \frac{rra + a - r \times yy}{rra}$: Whence

the Resistance of the Base to the Resistance of the convex Surface, is as yy to $2G$; and when $y = QR$,

it will be as $a - r$ to $\frac{ar}{a - r} \times \text{Log} : \frac{a}{r} : -r$.

Ex. 6.

Let ABE be an Hyperboloid; denoting the Quan- 210.

ties as in the last Example, $uu = aa + \frac{a}{r}yy$, and

X x

yy

FIG. 210. $\frac{yy}{1 + \frac{\dot{x}^2}{y^2}} = \frac{rra + ryy}{rra + r + a \times yy} yy$, whose Fluent $G =$

$$\frac{1}{2} \times \frac{ryy}{r+a} + \frac{1}{2} \times \frac{rraa}{r+a} \times 2.30258 \text{ Log. } \frac{rra + a + r \times yy}{rra}$$

Therefore Resistance of the Base : Resistance of the convex Surface :: yy to $2G$.

Ex. 7.

210. Let ABE be a Paraboloid, $AE = x$, $BE = y$
 $2rx = yy$, then $\dot{x}^2 = \frac{y^2 \dot{y}^2}{rr}$; and $\frac{yy}{1 + \frac{\dot{x}^2}{y^2}} =$

$$\frac{rryy}{rr+yy}, \text{ whose Fluent } G = 2.3025rr \text{ Log : } \sqrt{rr+yy}$$

$$\text{And by Correction } G = 2.302585rr \times \text{Log : } \frac{\sqrt{rr+yy}}{r}$$

And the Resistance of the Base, to the Resistance of the Surface of the Solid, is as yy to $2.30258rr$

$$\text{Log : } \frac{rr+yy}{rr} :$$

Ex. 8.

210. Suppose AB to be a cubic Parabola, $r^2x = y^3$
 then $\dot{x}^2 = \frac{9y^4 \dot{y}^2}{r^4}$, whence $\frac{yy}{1 + \frac{\dot{x}^2}{y^2}} = \frac{r^4 yy}{r^4 + 9y^4}$

$$\text{whose Fluent } G \text{ (by Form the 5th)} = \frac{.01745rr}{6}$$

$$\times \text{Degrees in the Arch whose Tangent is } \frac{3yy}{rr}$$

and the Resistance of the Base : Resistance of the Surface :: as yy to $2G$.

Ex. 9.

212. Let $ABDQ$ be the Solid generated by the Cycloid ABD revolving round AQ . $DQ = a$, $AE = x$,
 $EB = y$

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$EB = y$, $AB = z$; by the Nature of the Curve, $y = F I G$.

$z - \frac{zz}{4a}$, and $\dot{y} = \dot{z} - \frac{z\dot{z}}{2a}$; then $\frac{y\dot{y}}{1 + \frac{\dot{z}^2}{y^2}}$ 212.

$$= \frac{y\dot{y} \times \dot{y}^2}{\dot{z}^2} = \frac{y\dot{y} \times \overline{2a - z}^2}{4aa} = \frac{\overline{a - y} \times y\dot{y}}{a}, \text{ whose}$$

Fluent $G = \frac{y\dot{y}}{2} - \frac{y^3}{3a}$: Whence the Resistance of the Base, to the Resistance of the Surface :: as $y\dot{y}$, to $y\dot{y} - \frac{2y^3}{3a}$, or as 1 to $1 - \frac{2y}{3a}$, and in the whole Solid it is as 3 to 1.

P R O B. XXIII.

To find the Center of Gyration of any Body.

This Center is such a Point O , that a given Force acting at a given Distance from the Axis of Rotation SR , will in the same Time generate the same angular Velocity in that Body, as it would do if the whole Body was placed in O . 239.

Let A, B, C , &c. be any Particles of the Body, and let fall the Perpendiculars Aa, Bb, Cc , &c. and Or, Pp , on the Axis of Motion SR . And suppose a given Force S apply'd at P , acting at the given Distance Pp from that Axis, to move the Body about SR . The absolute Motion of the Particle A being as $Aa \times A$, the Force acting at A that generates it must also be as $Aa \times A$; and a Force acting at P that will generate the same Motion in A , is as $\frac{Aa}{Pp} \times Aa \times A$,

or $\frac{Aa^2 \times A}{Pp}$. Likewise the Forces at P that gene-

X x 2

rates

F I G.

239.

rates the Motions of B and C , will be as $\frac{Bb^2 \times B}{Pp}$

and $\frac{Cc^2 \times C}{Pp}$, &c. and the Sum of all these = S .

And the Force at P that can generate the same angular Motion of A , placed in A will be as $\frac{Or^2 \times A}{Pp}$

likewise the Force at P that can generate the Motion of the whole Body, placed in O , will be $\frac{Or^2 \times \text{Body}}{Pp}$

= S , by the Hypothesis. Therefore the Sum of all

the $\frac{Aa^2 \times A}{Pp} + \frac{Bb^2 \times B}{Pp} + \frac{Cc^2 \times C}{Pp}$, &c. =

$\frac{Or^2 \times \text{Body}}{Pp}$, whence $Or^2 = \frac{Aa^2 \times A + Bb^2 \times B + \&c.}{\text{Body}}$

Therefore if $z = \text{Body}$, $x = \text{Distance of any Parti-}$

cle, then $rO^2 = \frac{\text{Sum of } xxxz}{z} = \frac{\text{Fluent of } xxxz}{\text{Body}}$

Whence this

R U L E.

Multiply the Fluxion of the Figure by the Square of the Distance of the generating Point, Line, or Surface, from the Axis of Motion, and find the Fluent; which divided by the whole Body, the Square Root of the Quotient, is the Distance of the Center of Gyration.

EXAMPLE. I.

240. Let SA be a right Line moving round S . Let $SB = x$,

$SA = a$, then $\dot{z} = \dot{x}$, and $\text{Fl. } x^2 \dot{x} = \frac{x^3}{3}$, and $z = x$,

then $\sqrt{\frac{x^3}{3x}}$ or $x\sqrt{\frac{1}{3}}$ or $a\sqrt{\frac{1}{3}} = \text{Distance of the}$

Center of Gyration SO ; the same is true of a slender Cylinder.

Ex. 2.

Ex. 2.

In the Periphery of a Circle revolving about the Diameter *SK*. Let $CA=r$, $BA=x$, $SA=z$, c = Circumference, then $\dot{z} = \frac{r\dot{x}}{\sqrt{rr-xx}}$. And the whole

Fluent of $\frac{rx^2\dot{x}}{\sqrt{rr-xx}}$ (by Form 17) = $\frac{rrz}{2} = \frac{1}{4}c \times$

$\frac{rr}{2}$; and for the whole Circumference it is $\frac{crr}{2}$,

then $\sqrt{\frac{crr}{2c}} = r\sqrt{\frac{1}{2}} = CO$, the Distance of the Center of Gyration.

Ex. 3.

For the Plane of a Circle, or a Cylinder, revolving round the Axis in *C*. Let $CA=r$, $CB=x$, $c=3.1416$, then Circumference at $B = 2cx$, and $\dot{z} = 2cx\dot{x}$, and

Fluent $2cx^3\dot{x} = \frac{1}{2}cx^4$; and for the whole Circle, 'tis $\frac{1}{2}cr^4$; but the Area of the Circle = crr , whence

$\sqrt{\frac{cr^4}{2crr}} = r\sqrt{\frac{1}{2}}$, the Distance CO .

Ex. 4.

For the Plane of the Circle about the Diameter *RS*. Let $DC=r$, $CB=x$, then Flux. Figure = $\dot{x}\sqrt{rr-xx}$,

and the whole Fl. of $x^2\dot{x}\sqrt{rr-xx}$, when $x=r$, is $\frac{1}{4}rr \times$ Quadrant; and for the whole Circle, $\frac{1}{4}rr \times$ Circle, this divided by the Circle gives $\frac{1}{4}rr$, and $\sqrt{\frac{1}{4}rr}$ or $\frac{1}{2}r$ = Distance CO .

Ex. 5.

For the Surface of a Sphere about the Diameter *RS*. Let $DC=r$, $CB=x$, $RD=s$, $c=3.1416$. Then the

Circumference of the Circle described by D is = $2cx$,

and $2cx\dot{s}$ = Fluxion of the Figure = $\frac{2cx \times r\dot{x}}{\sqrt{rr-xx}}$, multiply

FIG.

multiply by xx and we have $\frac{2crx^3\dot{x}}{\sqrt{rr-xx}}$; but when

$x=r$, whole Fluent of $\frac{xx\dot{x}}{\sqrt{rr-xx}} = r$, and (by Form

17) the whole Fluent of $\frac{2crx^3\dot{x}}{\sqrt{rr-xx}} = \frac{4}{3}cr^4$, and for

the whole Surface tis $= \frac{8}{3}cr^4$, and the Surface of the

Sphere $= 4crr$, whence $\frac{8cr^4}{3 \times 4crr} = \frac{2rr}{3}$, and $\sqrt{\frac{2}{3}rr}$

or $r\sqrt{\frac{2}{3}} = CO$ the Distance required.

Ex. 6.

243. Let RAS be a Globe revolving about the Diameter RS ; let $CD=r$, $CB=x$, $c=3.1416$; then $2cx \times 2DBx\dot{x}$ or $4cx\dot{x}\sqrt{rr-xx} = \dot{z}$ = Fluxion of the Solid; and the Fluent of $4cx\dot{x}\sqrt{rr-xx} = \frac{8}{15}crr \times r^3$ (by Form 17) $= \frac{8}{15}cr^5$, when $x=r$. And the Globe $= \frac{4}{3}cr^3$; whence $\sqrt{\frac{2}{3}rr}$ or $r\sqrt{\frac{2}{3}} =$ Distance CO .

Ex. 7.

244. Let ACB be a Cone revolving about the Axis AB . Let $AB=a$, $BC=b$, $BE=x$, $c=3.1416$; then $DE = \frac{ab-ax}{b}$. And $\dot{z} = 2cx\dot{x} \times \frac{ab-ax}{b}$, and multiplying by xx , we have $2cax^3\dot{x} - \frac{2cax^4\dot{x}}{b}$, whose Fluent is $\frac{cax^4}{2} - \frac{2cax^5}{5b} = \frac{1}{10}cab^4$, when $x=b$. The Solidity of the Cone is $= \frac{1}{3}cabb$, then $\sqrt{\frac{2cab^4}{10cabb}} = b\sqrt{\frac{2}{10}} = BO$.

Ex. 8.

245. Let AD be a Paraboloid revolving about the Axis AB . Put $AB=a$, $BC=b$, $AF=s$, $FD=x$, $c=3.1416$, and $rs=xx$. Then $DE = a - \frac{xx}{r}$; and $\dot{z} = 2cx\dot{x}$

x

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FIG.
245.

$x a - \frac{xx}{r}$, which multiply'd by xx is $2cax^3x - \frac{2cx^5x}{r}$, whose Fluent is $\frac{cax^4}{2} - \frac{cx^6}{3r} = \frac{1}{2}cab^4$ when $x=a$: And Paraboloid $= \frac{1}{2}cabb$, therefore $\frac{2cab^4}{6cabb} = \frac{1}{3}bb$, and $\sqrt{\frac{1}{3}bb} = b\sqrt{\frac{1}{3}} = \text{Distance } BO$.

P R O B. XXIV.

To find the lateral Strength of a Piece of Timber, whose Section is any Figure given.

Suppose a Beam FA fixt with one End in a Wall, and a Weight P suspended at the other End F to break it, and let ABD be the Section of it where it breaks. Let the absolute Strength of 1 Fibre of the Wood be 1, and put $AB = a$, $BD = b$, $AC = x$, $CE = y$, $BC = v$, and $BF = 1$. When the Beam breaks, the Parts at B don't separate at all, and the Parts at A separate the first, and the Degree of stretching at any Plane C will be as BC . But by Mechanics the Resistance any Particle C makes, will be as the Degree of stretching, and therefore as BC . Hence

$a : 1 :: v : \frac{v}{a}$ = the Resistance or Tension of a Fibre at C ; and by reason of the bended Leaver ABF , whose Fulcrum is in B , $1 : v :: \frac{v}{a} : \frac{vv}{a}$ the effect of that Tension upon the Weight P ; or the Weight suspended at F to ballance that Resistance; and therefore $\frac{vvy}{a} = \text{Weight at } F \text{ to ballance the Resistance of all the Fibres}$

246.

F I G.

246.

bres in the Line CE ; that is $\frac{vy}{a}$ = Strength of all the Particles in CE . Therefore the Sum of all the $\frac{vvy}{a}$, in the Figure, or the Fluent of $\frac{vvyx}{a}$ or $\frac{yv^2\dot{v}}{a}$ will = whole Weight P or the Strength of the Beam.

R U L E.

Expunge v or y or $\dot{x}$ by the Nature of the Figure, and find the Fluent of $\frac{yv\dot{v}\dot{x}}{a}$ or of $\frac{yv^2\dot{v}}{a}$; and that will be as the Strength of the Beam.

COR. If that Fluent be divided by the Section (or Fl. $\dot{x}$ or $y\dot{v}$), the Quotient will show the Distance from B , where the total Strength of all the Fibres of the Beam being collected, it will be equally strong.

EXAMPLE I.

246. If ABD be a Parallelogram; then $y=b$, and $\frac{yv^2\dot{v}}{a}$
 $= \frac{bv^2\dot{v}}{a}$; whose Fluent is $\frac{bv^3}{3a} = (\text{when } v=a)$
 $\frac{1}{3}baa$, for the Strength, $= \frac{1}{3}a \times \text{Section}.$

E x. 2.

Let ABD be a Triangle; then $y = \frac{bx}{a}$, and $v = a - x$; whence $\frac{yv^2\dot{x}}{a} = \frac{bx\dot{x}}{aa} \times \frac{aa - 2ax + xx}{a}$.
 And the Fluent is $\frac{bax^2}{2a} - \frac{2bx^3}{3a} + \frac{bx^4}{4aa} = \frac{1}{12}ba^3$,
 when $x=a$, for the Strength, $= \frac{1}{12}a \times \text{Section}.$

If the Vertex be at B , then $y = \frac{bv}{a}$, and $\frac{yv^2\dot{v}}{a}$
 $=$

$= \frac{bv^3\dot{v}}{aa}$, and the Fluent $= \frac{bv^4}{4aa} = \frac{1}{4}baa$, when $v=a$, for the Strength, or $\frac{1}{4}a \times$ Section = Strength.

Ex. 3.

If *AEB* be a Circle; and $y = \sqrt{ax - xx} = \sqrt{av - vv}$,
and $\frac{yv^2\dot{v}}{a} = \frac{v^2\dot{v}}{a} \sqrt{av - vv}$; but $F : v \sqrt{av - vv}$

or $v^{\frac{1}{2}}\dot{v} \sqrt{a-v} =$ Semicircle: And by Form 17 the
whole Fluent of $v^2\dot{v} \sqrt{av - vv}$, or $v^{\frac{5}{2}}\dot{v} \sqrt{a-v} =$
 $\frac{1}{16}aa \times$ Semicircle, and when doubled, $F : \frac{yv^2\dot{v}}{a} =$

$\frac{1}{16}a \times$ Circle $= \frac{5 \times 3.1416}{16 \times 4} a^3$, for the Strength of
the whole circular Section $= \frac{5}{16}a \times$ Section.

Ex. 4.

In a hollow Cylinder, or the Periphery of the Circle
AEB. Let the Arch $= z$, then $y\dot{v} = \dot{z}$, and $\dot{z} =$

$\frac{\frac{1}{2}a\dot{v}}{\sqrt{av - vv}} = \frac{1}{2}a \times \frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{a-v}}$, therefore Fl. of $\frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{a-v}}$
 $= \frac{z}{\frac{1}{2}a}$. And by Form 17, whole Fluent of $(\frac{yv^2\dot{v}}{a})$

or $\frac{1}{2} \times \frac{v^2\dot{v}}{\sqrt{av - vv}}$ or $\frac{1}{2} \times \frac{v^{\frac{5}{2}}\dot{v}}{\sqrt{a-v}}$, is $= \frac{3aa}{8a} \times$ Se-
micircle, when $v=a$; and for the whole Periphery =
 $\frac{1}{2}a \times$ Circle $= \frac{3 \times 3.1416}{8} a^2$, the Strength; or $\frac{3}{8}a \times$
Section = the Strength.

The Problems delivered in this Section are exceed-
ing general, each of them comprehending an infi-
nite Number of particular Cases, and are sufficient
Y y here

F I G. here to shew the Method of investigating general Problems by the Method of Fluxions. I shall now proceed to exemplify the same Doctrine, in the Resolution of a few particular Problems, belonging to Physicks or Natural Philosophy; and the rather because this Sort of Problems has not been so common among the Writers of Fluxions.



S E C T

S E C T. III.

The Solution of Physical Problems, or such as occur in the Phænomena of Nature.

P R O B. I.

To find the Relation of the Fluxions, of the Times, Velocities, and Spaces described by Bodies in Motion; being acted upon by any accelerating Forces.

L E T b = Body or Quantity of Matter.
 m = Motion generated.
 F = Accelerative Force.
 t = Time of moving.
 v = Velocity.
 s = Space described.

Let t' = Moment of Time, or an exceeding small Part of Time, s' = Moment of Space, v' = Moment of Velocity, m' = Increment of Motion. Then whatever be the Law of the accelerating Force F , it may

F I G. be considered as uniform for the Moment of Time

i . Then by the common Mechanics, the Velocity generated in the Time i will be as the Force and the Time directly, and Quantity of Matter reciprocally;

that is $\dot{v} \propto \frac{Ft}{b}$. Likewise the Velocity may be look'd upon as uniform, for the Moment of Time

i ; therefore by the Laws of uniform Motion, the Space described will be as the Time and Velocity,

or $s \propto vt$. Also the Motion generated is as the

Force and Time, that is $\dot{m} \propto Ft$. Now s , t , $\dot{v}$, $\dot{m}$, being supposed infinitely small; substitute the Fluxions instead of the Moments, and we have $\dot{m} \propto Ft$,

$\dot{v} \propto \frac{Ft}{b}$, and $s \propto vt$, universally. And since by

Mechanics m is as bv , therefore $\dot{m} \propto b\dot{v}$. Hence from these ~~Equations~~ we shall have,

$$1. \dot{m} \propto Ft \propto b\dot{v}, \text{ and } i \propto \frac{\dot{m}}{F}.$$

$$2. s \propto vt, \text{ and } i \propto \frac{s}{v}.$$

$$3. \dot{v} \propto \frac{Ft}{b}, \text{ and } i \propto \frac{b\dot{v}}{F}.$$

$$4. v\dot{v} \propto \frac{Fs}{b}, \text{ and } s \propto \frac{bv\dot{v}}{F}.$$

And in any of these Forms, if b or F be constantly the same; then such constant Quantity or Quantities must be left out. Likewise if the Velocity or the Space be decreasing, you must write $-\dot{v}$, or $-s$ instead of $\dot{v}$, s .

COR. Hence, put $b = 16\frac{1}{2}$ Feet, the Space descended by b , in one Second; then $2b =$ Velocity acquired in one Second. Then will

* general Proportions

$$\left. \begin{aligned} \dot{v} &= \dot{s} \text{ or } 2bt \\ \dot{s} &= \dot{v} \text{ or } 2bt \\ \dot{i} &= \frac{\dot{s}}{2b} \text{ or } \frac{\dot{v}}{2b} \end{aligned} \right\} \text{when } v=2b, \text{ and } F=b; \text{ which is} \\ \text{the Case of falling Bodies.}$$

For the Velocity generated is as the Time, therefore $\dot{i}$ (Time) : $2b$ (Velocity) :: $\dot{t}$: $\dot{v}$:: $\dot{t}$: $\dot{v}$, and $\dot{v} = 2b\dot{t}$.

Likewise the Space described with the uniform Velocity $2b$, is as the Time; therefore, $2b$ (Space) :

$\dot{i}$ (Time) :: $\dot{s}$: $\dot{t}$:: $\dot{s}$: $\dot{t}$, and $\dot{s} = 2b\dot{t}$.

These Forms are general, and will be found serviceable in the Solution of many physical and mechanical Problems.

P R O B. II.

To find the Motion of a musical String, vibrating at very small Distances.

1. Let AB be the String, and let it be drawn to C , 214.
and there let go; now since the Force to move the Point C , by Mechanics, is as the Sine of the Angle ACn , or as that Angle itself when it is very small; therefore the Point C alone will first begin to move, and presently by the Flexure of the String in d and e these Points will also begin to move, and then the next Points to these, and so forward. Now by reason of the great Flexure in C , that Point will at first be very swiftly moved; and the Curvature in d and e being thereby increased, these Points are continually accelerated; and the Curvature in C being diminished, its

F I G. its Motion will be less accelerated. And universally
 214. these Points that are too slow being more accelerated,
 and those too swift being less accelerated, it will come
 to pass that, the Forces being at length rightly
 adjusted, all the Points of the String will acquire
 such Motions, as to be carried to the Axis together;
 and will continue to go and return together *ad infinitum*.

2. Now that this may be regularly performed, the
 String must always have the Form of the Curve
 AFXB, whose Nature is such, that the Angle of
 Contact, or the Curvature in any Point *F*, will be as
 the Ordinate *FE*; for then the Force at *F* being as
 the Curvature that is as *FE*, the Velocity generated
 will also be as *FE* the Space to be described; and the
 constant Accelerations and Velocities, and the Parts
 of the Ordinate described, and those to be described,
 will be as the wholes; and consequently any cor-
 respondent Parts of the Ordinates, and therefore the
 whole Ordinates, will be described in equal Times.

3. To find the Radius of Curvature; Let *AB* or
 2 *AZ* = *a*, *ZX* = *b*; *AE* = *x*, *EF* = *y*, *AF* = *z*, *e* = Radius
 of Curvature in the middle Point *X*. Let $\dot{z}$ be given;
 and by Prob V. Sect. II. the Radius of Curvature in *F*

= $\frac{\dot{z}y}{\dot{x}}$; therefore by the Nature of the Curve *y* : *b*

:: *e* : $\frac{be}{y} = \frac{\dot{z}y}{\dot{x}}$, and *ebx* = $\dot{z}yy$, and the Fluent is

ebx = $\frac{yy\dot{z}}{2}$, but in *X*, $\dot{x} = \dot{z}$, and *y* = *b* : Therefore

the Fluent corrected is *ebx* — *ebz* = $\frac{yy - bb}{2} \dot{z}$, and *ebx*

= *ebz* + $\frac{y - bb}{2} \dot{z} = \frac{2eb + yy - bb}{2} \sqrt{\dot{x}^2 + \dot{y}^2}$, and by Re-

duction, $\dot{x} = \frac{2eb + yy - bb \times \dot{y}}{\sqrt{4eb^3 - 4eby^2 - y^4 - b^4 + 2bby}}$

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F I G.

214.

= (because e is vastly greater than b or y) $\frac{2eb\dot{y}}{\sqrt{4eb^3 - 4eb\dot{y}y}}$

= $\frac{\dot{y}\sqrt{eb}}{\sqrt{bb - yy}}$. Whence (by Form the 10th) the

Fluent $x = \sqrt{eb} \times$ Arch whose Sine is $\frac{y}{b}$, Radius = 1;

and when $y = b$, then $x = \frac{1}{2}a$, and then $x = \sqrt{eb} \times$

$\frac{3.1416}{2} = \frac{c}{2}\sqrt{eb}$, putting $c = 3.1416$. Therefore

$\frac{1}{2}a = \frac{c}{2}\sqrt{eb}$, whence $e = \frac{aa}{ccb}$, the Radius of Cur-

vature in X . Therefore the Radius of Curvature in

$$F = \frac{aa}{ccy}.$$

4. To find the Motion of any Particle of the String as suppose of X the middle Point. Let $p =$ Tension of the String, or the Force that extends it; $w =$ Weight of the String; $Iz = x$, $v =$ Velocity

in I , $t =$ Time of describing XI ; $\dot{x}$, $\dot{v}$, $\dot{t}$, the Moments of x , v , and t . The Radius of Curvature in

I is $\frac{aa}{ccx}$. By Example 16, Prop. XIII. The

Force wherewith any Particle of the Curve at I is urged; is to the Tension of the String (p) :: as that

Particle ($\dot{z}$) : to the Radius of Curvature in I ($\frac{aa}{ccx}$) :

therefore the Force acting at $I = \frac{pccx\dot{z}}{aa}$. And hence

the Force upon the Particle $\dot{z}$ is as the Distance x :

Now by Mechanics or the Laws of Motion, the Ve-

locity $\times$ Moment of Velocity $\propto \frac{\text{Moment of Space} \times \text{Force}}{\text{Weight}}$

(since the Weight is as the Matter); which is a universal

E I G. verfal Proportion for these Quantities. (See Prob. I.)

214.

Now it is known that any heavy Body falling thro' $16\frac{1}{2}$ or f Feet gains a Velocity of $2f$ in 1 Second;

therefore x' = Velocity generated by that Body in fal-

ling thro' x with that Velocity $2f$, because the Velocities generated are as the Times, or as the Spaces

($2f$ and x') uniformly described with the given Velocity $2f$. Therefore, in this Case of falling Bodies, we

have the Velocity $\times$ Moment of Velocity = $2fx'$: and

$$\text{likewise } \frac{\text{Force} \times \text{Moment of Space}}{\text{Weight}} = \frac{px'}{p} = x'.$$

Lastly in the Case of the vibrating String $vv' =$ Velocity $\times$ Moment of Velocity. And, because the

String is homogeneous $a : n :: z' : \frac{nz'}{a} =$ Weight of z'

and $x' =$ Moment of Space. Therefore (by the Rules, Prop. XIII.) we get this Analogy (from the

general Proportion before laid down) $2fx' : x' :: vv' :$

$$\frac{-x' \times pccx'z'}{aa \times \frac{nz'}{a}}. \text{ Whence } vv' = \frac{-2fpccxx'}{na}, \text{ or } vv' =$$

$$\frac{-2fpccxx'}{na}; \text{ whence the Fluent is } vv = \frac{-2fpccx^2}{na}$$

But in X , $v=0$, $x=b$; therefore the Fluent corrected (by Prop. XII.) is $vv = \frac{2fpcc}{na} \times \overline{bb - xx}$

$$\text{And in } Z, \text{ where } x=0, v = bc\sqrt{\frac{2fp}{na}}, \text{ the Feet}$$

described in a Second.

5. Lastly

5. Lastly for the Time. Since the Moment of the F 1 G.

Time $\propto \frac{\text{Moment of Space}}{\text{Velocity}}$, universally. And in a ^{214.}

falling Body, $2f(\text{Space}) : 1 \text{ Second (Time)} :: \dot{x}(\text{Space}) :$

$\frac{x}{2f} = \text{Time of describing } x \text{ by the falling Body.}$

And likewise $\frac{\dot{x}}{2f} = \frac{\text{Space}}{\text{Velocity}}$; therefore from

the general Analogy, we get $t' = \frac{-\dot{x}}{v}$, or $t =$

$$\frac{-\dot{x}}{\sqrt{\frac{2fp}{na}} \times \sqrt{bb - xx}}, \text{ then the Fluent } t = \frac{-A}{\sqrt{\frac{2fp}{na}}}$$

(putting $A = \text{Arch whose Sine is } \frac{x}{b} \text{ and Radius } 1$).

And by Correction (for in X , $t = 0$, $x = b$;) $t =$

$$\frac{B}{\sqrt{\frac{2fp}{na}}} \text{ (putting } B = \text{Arch whose Cosine is } \frac{x}{b} \text{);}$$

and when $x = 0$, the whole Time $t = \sqrt{\frac{na}{2pf}}$; and

2t or the Time of one entire Vibration $= \sqrt{\frac{na}{2pf}}$, in Seconds.

COR. 1. Hence all the Vibrations great and small are performed in equal Times; for they are all expressed

by $\sqrt{\frac{na}{2pf}}$, in which b is not concerned.

COR. 2. The Number of Vibrations performed in one Second is $\sqrt{\frac{2fp}{na}}$.

COR. 3. Hence the Square of the Time of Vibration of any musical String, is as it's Length and Weight directly, and it's Tension reciprocally. And
Z z therefore

F I G. therefore in the same String and Tension the Time is
214. as the Length.

S C H O L I U M.

To construct the *harmonical Curve AFX*; let s be the Arch of a Circle, whose Radius is b , or XZ ; d the Diameter of another Circle, whose Circumference is a or AB . Make AE or $x = \frac{ds}{b}$, and erect $EF=y$, the Sine of s ; and F is in the Curve.

For we found $\dot{x} = \frac{\dot{y}\sqrt{eb}}{\sqrt{bb-yy}}$, and $e = \frac{aa}{ccb}$,
therefore $\dot{x} = \frac{a\dot{y}}{c\sqrt{bb-yy}} = \frac{d}{b} \times \frac{b\dot{y}}{\sqrt{bb-yy}} =$ (by
the Nature of the Circle) $\frac{ds}{b}$; whence $x = \frac{ds}{b}$;

Therefore AFB is a mechanical Curve.

I took a virginal String $29\frac{5}{100}$ Inches long, and weighing $8\frac{6}{100}$ Grains; and fastning it to the Virginal, I stretched it with $8\frac{1}{2}$ Pound Weight Avoirdupoise; and causing it to vibrate, I found it to be Unison with the Note *Ela* in the Base (the Note below the *Cliff*): By this Problem it appears, that the String made 300 Vibrations in a Second of Time. This Experiment I made very accurately. However, by Reason of the Resistance of the Air, and the larger Vibrations that the String makes, it is probable that the Time is a little prolonged; and that the Number of Vibrations in a Second may be something less than is assigned by this Problem.

P R O B. III.

To find the Velocity of a Projectile at A moving F I G.
in any given Curve QAO about the Center of 215 .
Force S .

Let the Distance $SA=D$, SB the Perpendicular on the Tangent at $A=P$. Radius of Curvature $CA=R$, c = Velocity of the Body at A , e = Velocity of a Body in a Circle at the same Distance SA , and acted on with the same Force; take the infinitely small Arch Aa , and draw am , an , parallel to SA , CA .

Then by similar Triangles, $P:D::an:am::$ centripetal Force tending to C : to centripetal Force tending to S : : versed Sine of the Arch Aa : versed Sine of the Arch (described in the same Time) whose Radius is $SA::\frac{cc}{R}:\frac{ee}{D}$. Therefore $PR:DD::cc:ee$.

Or thus, by Prob. V. Sect. II. $R=\frac{D\dot{D}}{\dot{P}}$; therefore $cc:ee::P\times\frac{D\dot{D}}{\dot{P}}:DD$; that is $cc:ee::P\dot{D}:D\dot{P}$.

COR. I. In the Ellipsis and Hyperbola, the Square of the Velocity of a Projectile moving round the Focus : is to the Square of the Velocity of a Body moving in a Circle at the same Distance : : as the Projectiles Distance from the other Focus : is to the Semi-transverse.

For let $2r$ = Transverse, $2b$ = Conjugate; then (by Ex. 3d. and 5th, Prob. XIX. Sect. II.) $P =$

Z z 2

$$\frac{bD}{\sqrt{2rD+DD}},$$

FIG. 215. $\frac{bD}{\sqrt{2rD+DD}}$, and $\dot{p} = \frac{brD\dot{D}}{(2rD+DD)^{\frac{3}{2}}}$. Therefore

$$cc : ee :: \frac{bD\dot{D}}{\sqrt{2rD+DD}} : \frac{brD^2\dot{D}}{(2rD+DD)^{\frac{3}{2}}} :: 2r+D$$

: r .

COR. 2. The Velocity of a Body revolving in an Ellipsis round the Center, is to the Velocity of a Body in a Circle at the same Distance; as the Conjugate to that Line of Distance, to the Distance itself.

For (by Ex. 4. Prob. XIX. Sect. II.) $P = \frac{br}{\sqrt{rr+bb-DD}}$, and $\dot{p} = \frac{brD\dot{D}}{rr+bb-DD^{\frac{3}{2}}}$,

whence $cc : ee :: (\frac{br\dot{D}}{\sqrt{rr+bb-DD}} : \frac{brD^2\dot{D}}{rr+bb-DD^{\frac{3}{2}}}) ::$

$rr+bb-DD : DD$. And $c : e ::$ Conjugate of D : to D .

COR. 3. The Velocity of a Body moving in a Parabola about the Focus, is to the Velocity in a Circle at the same Distance :: as $\sqrt{2}$ to 1.

For let $r =$ Latus Rectum, then by the Nature of the Parabola, $P = \frac{\sqrt{rD}}{2}$, and $\dot{p} = \frac{\dot{D}}{4} \sqrt{\frac{r}{D}}$.

Whence $cc : ee :: (\frac{\dot{D}\sqrt{rD}}{2} : \frac{D\dot{D}}{4} \sqrt{\frac{r}{D}} ::) 2 : 1$.

PROB.

P R O B. IV.

To find the Velocity of a descending Body in any Place P , let fall from the given Point D towards the Earth or any attracting Body; being acted upon by a Force which is as any Power of its Distance CP from the Center.

FIG.
216.

Let the Earth's Radius $CA=r$, $CD=a$, $CP=x$, $DP=a-x$, whose Fluxion is $-\dot{x}$, t = Time of descending thro' DP , v = Velocity acquired by that Descent. F the Force at P , which let be as x^n .

By Mechanics, when the Body is given, it is universally $vv \propto -F\dot{x} \propto -x^n\dot{x}$; (see Prob. I.) and

the Fluent $v^2 \propto \frac{-x^{n+1}}{n+1}$. But in D , $v=0$, $x=a$,

therefore the Fluent corrected is $v^2 \propto \frac{a^{n+1} - x^{n+1}}{n+1}$.

But if $n = -1$, then $vv \propto \text{Log.} \frac{a}{x}$.

Now we must find the Value of v at A the Earth's Surface for some determinate Values of a and x , in order to turn the general Proportion into an Equation. Thus, it is known by Experiments, that a heavy Body descending through a Space s or $16\frac{1}{2}$ Feet will acquire a Velocity of $2s$ or $32\frac{1}{2}$ Feet in a Second of Time. Therefore writing $2s$ for v , r for x , $r+s$ for

a , we shall get this Analogy, $\frac{r+s^{n+1} - r^{n+1}}{n+1} :$

$$4ss : \frac{a^{n+1} - x^{n+1}}{n+1} : vv = 4ss \times \frac{a^{n+1} - x^{n+1}}{r+s^{n+1} - r^{n+1}} =$$

F I G. 216.
$$\frac{a^{n+1} - x^{n+1}}{n+1 \times r^n s} ; \text{ and } v = 2s \sqrt{\frac{a^{n+1} - x^{n+1}}{n+1 \times r^n s}},$$

Feet described in a Second.

COR. 1. If $n = -2$, then $v = 2r \sqrt{\frac{s \times a - x}{ax}}$, therefore the Velocity of a Body falling from an infinite Distance to the Surface of the Earth will be $2\sqrt{rs}$.

COR. 2. If $n = 0$, then $v = 2 \sqrt{s \times a - x}$.

COR. 3. If $n = 1$, then $v = \sqrt{\frac{2s}{r} \times aa - xx}$. Therefore a Body falling from the Surface of the Earth to the Center acquires the Velocity $\sqrt{2rs}$.

COR. 4. If the Body had been projected downwards from D with any Velocity c , then instead of $v = 0$, at D put $v = c$, and the corrected Fluent will be $vv - cc$ or $\frac{a^{n+1} - x^{n+1}}{n+1}$, or $vv - cc = 4s \times \frac{a^{n+1} - x^{n+1}}{n+1 \times r^n}$.

Or if the Body was projected upwards, $cc - vv$ or $\frac{x^{n+1} - a^{n+1}}{n+1}$, or $cc - vv = 4s \times \frac{x^{n+1} - a^{n+1}}{n+1 \times r^n}$.

and in either Case $vv = cc + 4s \times \frac{a^{n+1} - x^{n+1}}{n+1 \times r^n}$.

COR. 5. If n be equal or greater than -1 ; as $0, 1, 2$, &c. a Body falling from an infinite Distance will acquire an infinite Velocity: For then a will be infinite, and a^{n+1} will be in the Numerator. But if n be less than -1 ; as $-2, -3, -4$, &c. the Velocity acquired at the Surface, in falling from an infinite Height, will be $\sqrt{\frac{-4sr}{n+1}}$, for then a is in the Denominator and infinite, and x is $= r$.

P R O B

PROB. V.

To find the Time wherein a falling Body will descend through any Space towards the Earth, &c. being acted upon by a Force which is as any Power of its Distance from the Earth's Center. FIG. 216.

The same Things supposed as in the last Problem, we shall have, the Moment of Time $\propto \frac{\text{Mom. Space}}{\text{Velocity}}$, universally. Since a descending Body at the Earth's Surface acquires the Velocity $2s$ in the Time p or Second, therefore by the Laws of uniform Motion,

$2s : p :: \dot{x} : \frac{p\dot{x}}{2s} = \text{Moment of Time, wherein } \dot{x}$
is described with Velocity $2s$. Hence from the uni-

versal Proportion, $\frac{p\dot{x}}{2s} : \frac{\dot{x}}{2s} :: t : \frac{-\dot{x}}{v}$; there-

fore $t = \frac{-p\dot{x}}{v} = \frac{-\dot{x}}{v}$ (because $p=1$). And $t =$

$\frac{-\dot{x}}{v} = (\text{by the last Problem}) \frac{-\dot{x}}{2s\sqrt{\frac{a^{n+1}-x^{n+1}}{n+1 \times r^n s}}}$;

whence $t = \text{Fluent of } \frac{-\dot{x}}{2s\sqrt{\frac{a^{n+1}-x^{n+1}}{n+1 \times sr^n}}}$, expressed

in Seconds.

COR.

COR. 1. If $n = -2$, $t =$ Fluents of $-\frac{1}{2r} \sqrt{\frac{a}{s}} \times \frac{x}{\sqrt{a-x}}$

$=$ (by Form 10 and 11) $\frac{1}{2r} \sqrt{\frac{a}{s}} \times ax - xx - \frac{a}{2r} \sqrt{\frac{a}{s}}$
 $\times ,017453 \times$ Degrees in the Arch whose Sine is
 $\sqrt{\frac{x}{a}}$, and Radius 1. And when duly corrected

the Time $t = \sqrt{\frac{aax - axx}{4rrs}} + \frac{,017453a}{2r} \sqrt{\frac{a}{s}} \times$ De

grees in the Arch whose Cosine is $\sqrt{\frac{x}{a}}$, and Radi

us 1. And the Time of descending to the Center is

$$\frac{3.1416a}{4r} \sqrt{\frac{a}{s}}.$$

COR. 2. If $n = 0$, then $t = \frac{-x}{\sqrt{4s \times a - x}}$, and $t =$

$$\sqrt{\frac{a-x}{s}}.$$

COR. 3. If $n = 1$, then $t = \sqrt{\frac{x}{2s}} \times \frac{-x}{\sqrt{aa - xx}}$, and

(by Form 10) $t = -\sqrt{\frac{r}{2s}} \times ,017453$ Degrees of the

Arch whose Sine is $\frac{x}{a}$, and Radius 1. And being

duly corrected, $t = ,017453 \sqrt{\frac{r}{2s}} \times$ Number of De

grees in the Arch whose Cosine is $\frac{x}{a}$, and Radius 1.

Hence the Time of descending to the Center will be

$$\frac{3.1416}{2} \sqrt{\frac{r}{2s}} : \text{And therefore all the Times of Def}$$

cent from any Altitudes whatsoever will be equal.

SCHOLIUM.

If t be given to find x ; find the Fluent of $\dot{t} =$
 $-\frac{1}{2}r\sqrt{\frac{a}{s}} \times \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a-x}}$, or $\sqrt{\frac{2r}{s}} \times \frac{-\dot{x}}{\sqrt{aa-xx}}$, by
 infinite Series and revert the Series. And if either t or
 x be given the other may be found by first finding $\dot{x}$.
 And hence a Body being projected upwards or down-
 wards with any Velocity; its Height may be found,
 and the Time of its Ascent, or Descent. For then $\dot{t} =$
 $\frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{cc+4s \times \frac{a^{n+1}-x^{n+1}}{n+1 \times r^n}}}$; by Cor. 4. Pr. 4:

P R O B. VI.

*The Velocity and Direction of a Projectile, and the Law
 of centripetal Force being given; to find the Velocities,
 Times and Angles of Revolution.*

Let C be the Center of Force, and let the Body be
 projected from V in Direction VA with Velocity b
 describing the Space b in the Time g ; and let p be the
 Velocity and q the Space which the Force at V will
 generate in the same Time g . To the Center C de-
 scribe the Circle VXH , draw the Radii CX , CV in-
 finitely near, cutting the Trajectory in I and K , and
 describe the Arch Kr . Call CV , a ; CI , x ; VI , u ;
 VX , z ; Ir , $\dot{x}$; Kr , $\dot{y}$; IX , $\dot{z}$; IK , $\dot{u}$: And let $t =$
 Time of describing VI , and $\dot{t}$ of describing IK , $s =$
 Sine, $c =$ Cosine of the Angle CVA , and let the
 Force

F I G. Force in any Place I be as x^n , and v = Velocity in L
 217. Then.

1. By the Resolution of Forces, the Force to accelerate the Body in Direction of the Curve is $x^n \times$

$$\frac{Ir}{IK} = x^n \times \frac{-x'}{u}; \text{ but by Mechanics (see Prob. I.)}$$

$$vv' \propto \text{Force} \times u, \text{ universally. Whence } vv' \propto \frac{-x'x}{u}$$

$\times u \propto -x'x$. Hence therefore the Moment of Velocity depends not at all on the Angle KIC , but upon x the Moment of perpendicular Descent, and is therefore the same at all Inclinations as if the Body descended perpendicularly. Now to find the Moment of Velocity in V ; by the Laws of uniformly accelerated Motion, the Velocity generated is as the Time, or as the Space uniformly described with a given Velocity: And since in the Time of describing $2q$, the

Velocity p is generated, therefore $2q : p :: x' : \frac{px'}{2q}$
 $=$ Moment of Velocity generated at V whilst x is described with Velocity p . Here therefore the Value of

$$vv' \text{ is } \frac{ppx'}{2q}, \text{ and Force} \times \text{Moment of Space is } a^n x$$

Therefore from the universal Proportion, $\frac{ppx'}{2q} : a^n x$
 $:: vv' : -x^n x' :: vv' : -x^n x$, whence $vv' = \frac{-ppx'x}{2qa^n}$

and the Fluent $v^2 = \frac{-ppx^{n+1}}{n+1 \times qa^n}$. But in V , $x=a$,

$$\text{and } v=b, \text{ therefore by Correction } vv = bb + \frac{ppa}{n+1 \cdot q}$$

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F I G.
217.

$\frac{ppx^{n+1}}{n+1 \cdot qa^n}$. But here if $n = -1$, then $vv = bb +$

$$\frac{ppa}{q} \times \text{Log. } \frac{a}{x}.$$

2. Again, since the Velocity is every where reciprocally as the Perpendicular let fall on the Tangent; and these Perpendiculars in V and I are sa , and

$$\frac{xy}{u}; \text{ therefore } b : v :: \frac{xy}{u} : sa, \text{ then is } y = \frac{asb}{vx} u =$$

$$\frac{asb}{vx} \sqrt{x^2 + y^2}, \text{ and by Reduction } y = \frac{-asbx}{\sqrt{v^2x^2 - a^2s^2b^2}}.$$

But by similar Triangles $x : y :: a : z = \frac{ay}{x}$, and $\dot{z} = \frac{ay}{x}$, therefore $\dot{z} = \frac{-aasbx}{x\sqrt{v^2x^2 - a^2s^2b^2}}.$

3. Lastly, since $\dot{t} \propto \frac{xy}{2}$ the Moment of the Area, or the Time $\propto$ Area; and in V , the Area = $\frac{asb}{2}$; therefore as $g : asb :: \dot{t} : xy :: t : xy$, then will $\dot{t} = \frac{gxy}{asb}$, or $\dot{t} = \frac{-gbx\dot{x}}{b\sqrt{v^2x^2 - a^2s^2b^2}}.$

Consequently substituting for vv it's equal, in the Values of $\dot{z}$ and $\dot{t}$, the Fluents will give z and t .

COR. I. Hence the Apfides of the Trajectory are easily found; for then it will be $u = y = \frac{asb}{vx} u$, or

$$asb = vx; \text{ therefore } vv = \frac{a^2s^2b^2}{xx} = bb + \frac{ppa}{n+1 \times q}$$

$$\frac{ppx^{n+1}}{n+1 \times qa^n}.$$

Whence x will be had: And if

A a a 2

two

FIG. two Roots of this Equation be found; then the corresponding Fluents x will give the Position of the Apfides.

COR. 2. Since the Sine of the Angle $CIK = \frac{y}{u}$
 $= \frac{asb}{vx}$; therefore if that Angle be given, the Distance x may be found; or if x be given, the Angle may be found.

COR. 3. If the Body be projected at right Angles to CV , then $s = 1$, and, by Cor. 1, we shall have
 $bbxx + \frac{ppax^2}{n+1 \times q} - \frac{ppx^{n+3}}{n+1 \times qa^n} = a^2bb$; in which one Root is a ; and finding x another Root in the Equation, the Fluent x may thence be had; and consequently the Motion of the Apfides.

COR. 4. If in any Part of the Orbit, $vx = asb$, and v be greater than $\sqrt{\frac{ppx^{n+1}}{2qa^n}}$, the Body will fly off from the Center; but if v be less, it will approach the Center; if equal, it will move in a Circle.

For by the Laws of centripetal Force $\sqrt{2aq}$ will be the Arch described in the Time g , by a Body revolving in a Circle at the Distance a ; and $2q : p :: \sqrt{2aq} : \sqrt{\frac{ppa}{2q}} =$ the Velocity of a Body moving in a Circle at the Distance a . And $\sqrt{a^{n+1}} : \sqrt{x^{n+1}} :: \sqrt{\frac{ppa}{2q}} : \sqrt{\frac{ppx^{n+1}}{2qa^n}} =$ the Velocity of a Body moving in a Circle at the Distance x . Consequently when v is greater, lesser, or equal to $\sqrt{\frac{ppx^{n+1}}{2qa^n}}$, the Body will fly from, or approach the Center, or move in a Circle.

P R O B.

P R O B. VII.

To find the Time of a Body's descending thro' any Arch of a Cycloid.

Let AC be the Axis, BV , DF , ef , Ordinates. Let $F I G$;
 $AC = a$, $VC = b$, $VF = x$, $Ff = x$, $BD = z$, $De = z$,
 $s = 16\frac{1}{2}$ Feet. $t =$ Time of describing BD : And
 let the Body fall from B . 218.

The Times of describing any Spaces uniformly are as the Spaces directly, and the Velocities reciprocally; but the Velocities are as the Square Roots of the Heights fallen from; and $2s$ is the Space uniformly described in 1 Second, by the Velocity acquired by

falling through s : Therefore $\frac{2s}{\sqrt{s}} : 1 \text{ Second} ::$

$\frac{z}{\sqrt{x}} : t = \frac{z}{2\sqrt{sx}}$; or $t = \frac{z}{2\sqrt{sx}}$; but (by Ex. 8.

Prob. VIII. Sect. II.) $\dot{z} = \dot{x} \sqrt{\frac{a}{b-x}}$; whence $t =$

$\int \frac{a}{s} \times \frac{x^{-\frac{1}{2}} \dot{x}}{\sqrt{b-x}}$. Whence the Fluent (by Form

the 10th) is $t = \sqrt{\frac{a}{s}} \times \text{Arch whose Sine is } \sqrt{\frac{x}{b}}$, and

Radius 1. And when $x = b$, then $t = \frac{3.1416}{2} \sqrt{\frac{a}{s}}$.

Whence if a Pendulum be made to vibrate in the Arch of the Cycloid, $2t$ or the Time of one entire Vibration will be $3.1416 \sqrt{\frac{a}{s}}$, in Seconds.

COR.

FIG. COR. 1. Hence all the Times are equal in which
218. Bodies descending from any Points Z, B, D shall arrive at the lowest Point C : And all the Times of Vibration will be equal among themselves.

COR. 2. It appears by Ex. 3. Prob. VII. Sect. II. that if $ZP, P\mathcal{Q}$ be two Cycloids, whose Cusps are at P , and Vertices at Z and $\mathcal{Q}$; then if a Pendulum PC be suspended at P , so that in oscillating it may fold about the Curves $ZP, P\mathcal{Q}$; then the Point C will describe the Cycloid $ZC\mathcal{Q}$: And therefore the Time of its Vibration will be $3.1416 \times$ Time of a Body's falling through AC .

P R O B. VIII.

219. *To find the Force wherewith a Corpuscle P is attracted to the Plane of a Circle $E\mathcal{Q}D$, according to any Law of centripetal Force.*

Let A be the Center, and AP perpendicular to the Plane of the Circle, and let the Force of each Particle be as the n^{th} Power of the Distance.

Put $AP = a$, $AE = x$, $c = 3.1416$. And let the Body m attract the Corpuscle P , at the Distance d , with the Force f ; then the Force which any Particle

x attracts the Corpuscle P towards E is $\frac{fx'}{md^n} \times$

$\overline{aa+xx}^{\frac{n}{2}}$, and the Force of all the Particles in the

Periphery $E\mathcal{Q}D$ is $\frac{2cfxx'}{md^n} \times \overline{aa+xx}^{\frac{n}{2}}$: And by Me-

chanics the Force in Direction $PA = \frac{2cfaxx'}{md^n} \times$
 $aa+xx$

$\sqrt{aa+xx}^{\frac{n-1}{2}}$; and the Fluxion of the Force = $\frac{2caxx}{md^n}$

$\times \sqrt{aa+xx}^{\frac{n-1}{2}}$; whose Fluent is $\frac{2cfa}{md^n} \times \frac{\sqrt{aa+xx}^{\frac{n+1}{2}}}{n+1}$.

But in A , $x=0$; therefore by Correction, the Force exerted on P by the Plane of the Circle ED is =

$\frac{2cfa}{md^n} \times \frac{\sqrt{aa+xx}^{\frac{n+1}{2}}}{n+1} - \frac{2cfa^{n+2}}{n+1 \times md^n}$. But if $n =$

-1 , then the Force = $\frac{cfad}{m} \times 2.302585 \text{ Log.}$

$\frac{\sqrt{aa+xx}}{aa}$.

COR. 1. Hence if $n = 1$, the Force of the Circle exerted on the Particle P will be $\frac{cfaxx}{md}$, the same as if the said Circle were wholly collected into the Center A .

COR. 2. Therefore if $n = 1$, a Sphere will attract any Particle P with the same Force as if the whole Sphere was contracted into the Center C . For taking the Circles ED , ed parallel, and equidistant from the Center; the Sum of the Forces will be as the Sum of the two Circles each multiplied into its Distance, or as either Circle into half the Sum of the Distances, that is into $2PC$; or both Circles multiplied into PC ; and it is the same of all equidistant Circles that compose the Globe.

COR. 3. If $n = -2$, the Force = $\frac{2cfdd}{m} \times 1 - \frac{a}{\sqrt{aa+xx}}$

= $\frac{2cfdd}{m} \times 1 - \frac{PA}{PE}$.

COR. 4. If n be less than -1 , then the Force of the whole infinite Plane will be $\frac{2cfa^{n+2}}{md^n \times -n-1}$.

PROB. IX.

FIG. To find the Force wherewith an infinite Solid, plain on one Side Ll , attracts a Corpuscle placed at C : Supposing the Law of Attraction to be universally as some Power of the Distance greater than 1.

221.

Draw the infinite Line CGK , perpendicular to the Plane LGl , and through the Points I, K , infinitely near each other, draw two Planes parallel to Ll ;

and let $CG = a$, $CI = x$, $IK = \dot{x}$, and the Force as x^n ; and by Cor. 4. Prob. VIII. the Force wherewith the Solid contained between the Planes at I, K ,

attracts the Corpuscle, will be $\frac{2cfx^{n+2}\dot{x}}{md^n \times -n-1}$, and

its Fluxion $\frac{2cfx^{n+2}\ddot{x}}{md^n \times -n-1}$, whose Fluent is $\frac{2cf}{-md^n}$

$\times \frac{x^{n+3}}{n+1 \times n+3}$. And when duly corrected the Force

will be $\frac{2cf}{n+1 \times n+3 \times md^n} \times \frac{a^{n+3} - x^{n+3}}{n+1 \times n+3 \times md^n}$.

COR. 1. If n be less than -3 , and the Solid infinite towards K ; the Force will be $\frac{2cfa^{n+3}}{n+1 \times n+3 \times md^n}$.

COR. 2. Hence therefore the Force at different Distances from the infinite Solid (when n is less than -3) will be as a^{n+3} or CG^{n+3} .

COR. 3. Hence also (if n be less than -3), the Force of a very great Body upon a very small Particle,

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cle, at any extremely small Distances, will be as the $F I G.$
 $n+3^{th}$ Power of the Distance, nearly. 221.

COR. 4. And if the Corpuscle be placed within the Solid at H , so as $GH = GC$, the Force will be the very same as if it were placed at C , so far without it. For, taking $IH = GH$, the Solids HG and HI destroy one another's Effects.

PROB. X.

To find the Force wherewith a Sphere attracts a Corpuscle P , situated either without or within the Sphere; supposing the Forces of all the Particles to be reciprocally as the Squares of their Distances.

Case 1. Let P be without the Sphere. Draw the Axis PAB , and the Ordinates ED , ed infinitely near, and let S be the Center, put $PS = a$, $PD = x$, $Dd = x'$, $AS = r$, $ED = y$, $bb = aa - rr$. Then PE

222.

$= \sqrt{xx + yy} = \sqrt{xx + rr - a - x^2} = \sqrt{rr - aa + 2ax} = \sqrt{2ax - bb}$. But by Cor. 3. Prob. VIII. the Force of the infinitely thin Solid contained between the

Planes of the Circles DE , de is $= \frac{2scdd}{m} \times 1 - \frac{PD}{PE}$,

therefore the Fluxion of the Force is $\frac{2cfdd}{m} \times$

$\frac{xx}{\sqrt{2ax - bb}}$, whose Fluent is $\frac{2cfdd}{m} \times$

$x - \frac{bb + ax}{3aa} \sqrt{2ax - bb}$, but in A where the Force

is 0, $x = a - r$; therefore by proper Correction, the
 B b b Force

Force of the Segment EAF is $= \frac{2cfdd}{m} \times$ into:

$$x - a + r + \frac{bb + aa - ar}{3aa} \times a - r - \frac{bb + ax}{3aa} \times PE:$$

and when $x = a + r$, the Force of the whole Sphere is
 $= \frac{2cfdd}{m} \times \frac{2r^3}{3aa}.$

223. Case 2. Let P be within the Sphere. Let $PQ = b$,

$PD = x$, then $PE = \sqrt{rr - aa + 2ax} = \sqrt{bb + 2ax}$;
 therefore (by Cor. 3. Pr. VIII.) the Fluxion of the

Force at D is $\frac{2cfdd}{m} \times \dot{x} - \frac{x\dot{x}}{\sqrt{bb + 2ax}}$; whose

Fluent is $\frac{2cfdd}{m} \times x + \frac{bb - ax}{3aa} \sqrt{bb + 2ax}$: but

in P , $x = 0$; therefore the Force of the Zone or
 Section $QEDFR = x + \frac{bb - ax}{3aa} \sqrt{bb + 2ax} - \frac{b^3}{3aa}.$

In which writing $a + r$ for x , there comes out $\frac{2cfdd}{m} \times$

$\frac{r^3 + a^3 - b^3}{3aa}$ for the Attraction of the Segment $QBR.$

And by a like Process the Attraction of the Segment

$QAR = \frac{2cfdd}{m} \times \frac{r^3 - a^3 - b^3}{3aa}$: whose Difference

is $\frac{2cfdd}{m} \times \frac{2}{3}a$, the absolute Force of the Corpuscle

P towards the Center; which is the same as the Force
 of a Sphere, whose Radius is SP , acting on the Cor-
 puscle P at its Surface.

COR. 1. Hence the Force of the Sphere upon the
 Particle P placed without the Sphere is the very same
 as if the whole Sphere was collected into the Center,
 and exerted the Sum of all the Forces from that Cen-

ter. For $\frac{4r^3c}{3aa} =$ Sphere divided by the Square of
 the

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FIG.
223.

the Distance; and $\frac{m}{dd} : f :: \frac{4r^3c}{3aa} : \frac{4cr^3d^2f}{3ma^2}$ the very same Force of the Sphere before found. And hence it is also evident, that the Forces of Spheres are accurately in the reciprocal Ratio of the Squares of the Distances from their Centers.

COR. 2. The Force wherewith any Corpufcle P within a Sphere, is attracted to the Center, is accurately as its Distance from the Center; and is the same in different homogeneous Spheres, as long as that Distance is the same.

COR. 3. Hence also, the Forces of Gravity at the Surfaces of any homogeneous Spheres, are as the Radii of the Spheres. For the Force is as $\frac{2cfdd}{m} \times$

$\frac{2r^3}{3aa}$, that is as $\frac{r^3}{aa}$, and when $a = r$, it will be as $\frac{r^3}{rr}$ or as r .

P R O B. XI.

To find the Force wherewith a Spheroid attracts a Corpufcle P , lying upon its Surface in the Axis PB . 224

Let $PB = 2r$, Diameter $GH = 2a$, $AP = x$, then

$$AE = \frac{a}{r} \sqrt{2rx - xx}, \text{ and } PE = \sqrt{xx + \frac{aa}{rr} \times 2rx - xx}$$

$$= \frac{1}{r} \sqrt{2ra^2x + rrx - aaxx} = \frac{1}{r} \sqrt{2raax + bbxx},$$

putting $bb = rr - aa$. By Cor. 3. Prob. VIII. the

$$\text{Fluxion of the Force at } A \text{ is } = \frac{2cfddx}{m} \times 1 - \frac{PA}{PE}$$

$$B b b 2 =$$

$$224. = \frac{2cfd d}{m} \times : \dot{x} - \frac{rx^{\frac{1}{2}} \dot{x}}{\sqrt{2ra^2 + bbx}} : \text{Let } \phi = \frac{2.302585}{\frac{1}{2}b}$$

$$\times \text{Log. } \frac{b\sqrt{x + \sqrt{2raa + bbx}}}{\sqrt{2raa}}, \text{ if } r \text{ be greater than } a :$$

or $\phi = \frac{.017453}{\frac{1}{2}b} \times \text{Degrees in the Arch, whose Sine is } \frac{b}{a} \sqrt{\frac{x}{2r}}, \text{ if } r \text{ is less than } a ; \text{ and the Fluent is } = \frac{2cfd d}{m} \times : x + \frac{rraa}{bb} \phi - \frac{r}{bb} \sqrt{2ra^2 x + b^2 x^2} :$

$= \text{Force wherewith the Segment } PEF \text{ attracts the Particle at } P.$

COR. 1. Hence the Force wherewith the whole Spheroid attracts the Particle P is $= \frac{2cddf}{m} \times \text{into } 2r + \frac{r^2 a^2}{bb} \phi - \frac{2r^3}{bb} : \text{ where } \phi = \frac{2.302585}{\sqrt{rr - aa}} \times 2 \text{ Log} : \frac{r + \sqrt{rr - aa}}{a} : \text{ or } \phi = \frac{.017453}{\sqrt{aa - rr}} \times \text{twice the Number of Degrees in the Arch whose Sine is } \frac{\sqrt{aa - rr}}{a}, \text{ Radius} = 1, \text{ according as } r \text{ is greater or lesser than } a.$

COR. 2. If b be very small, the Force of the Segment $EPF = \frac{2cdd}{m} f \times \text{into } x - \frac{2rx}{3a} \sqrt{\frac{x}{2r}} + \frac{bbxx}{10a^3} \sqrt{\frac{x}{2r}} : \text{ very near. And the Force of the whole Spheroid is } = \frac{2cfd d}{m} \times \text{into } 2r - \frac{4rr}{3a} + \frac{2rr}{5a^3} bb ; \text{ as will appear by infinite Series, or Form the 16th.}$

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COR. 3. Let $APBQ$ be an Ellipsis, AB the greater F I G.

Axis, PQ the lesser. Put $\phi = \frac{2.30258}{\sqrt{AC^2 - PC^2}} \times 227.$

Log: $\frac{AC + \sqrt{AC^2 - PC^2}}{PC}$: then the attracting Force

at A upon the oblong Spheroid, generated by revolving round AB , is $2AC + \frac{2AC^2 \times PC^2}{AC^2 - PC^2} \phi - \frac{2AC^3}{AC^2 - PC^2}$.

And (by the last Prob.) the Force at A upon a Sphere whose Diameter is AB , is $\frac{2}{3}AC$. Therefore the Force of the Spheroid, to that of the Sphere, is as $AC \times \phi - 1 \times 3PC^2$, to $AC^2 - PC^2$. And since the oblate Spheroid generated by revolving about PQ , is a mean proportional between the oblong Spheroid and this Sphere; therefore the Force upon the Equinoctial at A , upon this oblate Spheroid (which represents the Earth) is nearly a mean between the Forces of these other two Bodies. And this may help to determine the Figure of the Earth:

P R O B. XII.

To find the Motion of a Ray of Light passing into a refracting Medium.

Let there be two Mediums separated by the refracting Space $RAdD$ terminated by the Parallel Planes RA , Dd ; and let the Ray, moving in the Direction GH , pass from H to I , and in its Passage be acted upon, in Lines perpendicular to the Planes, by any Force which is equal at equal Distances from either Plane, and at all Distances as any Powers or Sums of Powers of the Distance therefrom. And let

225.

F I G. let b = Velocity in H , v = Velocity in P , $CP = z$,

225. $PI = z$, $PF = z$, t = Time of describing PF . And let the Force be as $A + Bx^m + Cx^n + Dx^r \&c. = Q$,

then will the Force in Direction PF be $\frac{x}{z} Q$. And since

the Velocity $\propto \frac{\text{Space}}{\text{Time}}$, and Moment of Velocity $\propto$

Force $\times$ Time, therefore $vv \propto Qx$, or $vv \propto Qx$; assume p a given Quantity, and let $vv = pQx$, and let F be the Fluent of Qx , then $v^2 = 2pF$, and by Correction $vv - bb = 2pF$, and $vv = bb + 2pF$. Hence the Ray will always have the same Velocity in the same Medium DIK , whatever be the Angle of Incidence.

Let the Motion of the Ray GH be divided into two GA , AH , one parallel the other perpendicular to the Plane RA . Then since the parallel Motion AH is not at all changed by the Actions of the Forces perpendicular to these Planes: Therefore if ID be made $= AH$, and DK perpendicular to DI , then IK will be described in the same Time as GH . Therefore drawing IE parallel to GH , the Velocity in H to the Velocity in I , is as GH or EI to IK , that is as the Sine of the Angle of Refraction to the Sine of the Angle of Incidence. And therefore the Velocity of Light in Vacuo, to its Velocity in Air of a mean Density at the Surface of the Earth; is as $.999\frac{1}{3}$ to 1. For by Experiment the Sines of Refraction and Incidence are in that Ratio.

COR. I. *The Sine of the Angle of Incidence at one Plane, is to the Sine of the Angle of Emergence from the other Plane, in a given Ratio.* For the Velocity v at the second Plane will always be equal to the given Quantity $\sqrt{bb + 2pF}$, and the Sine of Incidence to the Sine of Emergence, as this given Quantity to b .

COR.

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COR. 2. If the Ray was to fall on I in Direction KI, F I G. with the Velocity it has at I, it would return in the same Curve IPH, and so go to G, and obtain its first Velocity. For the same Forces that did before accelerate its Passage, will now equally retard it in returning. Therefore,

COR. 3. If the Ray have a greater Velocity in the first Medium, than in the second, and the Angle of Incidence GHA be continually diminished, the Ray will at last be reflected; and the Angle of Reflection gba will be equal to the Angle of Incidence GHA. 226.

For let the Angle GHA be such, that the Ratio of its Cosine to the Radius may be equal or greater than the Ratio of the Sine of Incidence of the first Medium, to the Sine of Emergence in the second; and the Ray at R will be moving in a Direction parallel to the Planes; but being afterwards acted on by the same Forces as before, it will be turned back describing the Line Rbg similar and equal to RHG, and the Angle gba = GHA.

COR. 4. Hence if there be two similar Mediums whose Densities are p and q; and the Velocities after Refraction into each of them z and y: Then will $zz - bb : yy - bb :: p : q$.

For since the Forces of Attraction are made towards Bodies, these Forces will be proportional to the Causes that produce them, and therefore will be as the Densities of these Bodies, supposing the internal Form and Constitution of the Bodies to be in other Respects the same. The Forces therefore exerted at any equal Distances by these two Mediums will be as pQ and qQ ; whence will be had $zz - bb = 2pF$, and $yy - bb = 2qF$; whence $zz - bb : yy - bb :: p : q$, that is as the Densities of the Bodies, nearly.

COR. 5. If Light pass thro' several refracting Mediums, the Sum of all the Refractions will be equal to the single Refraction it would have suffered, by passing immediately out of the first Medium into the last.

For

FIG. For supposing these several Mediums to be separated
 226. by parallel Planes, the Refraction, Velocity, or Motion generated in approaching any one of these Mediums, will be destroyed again in its receding (on the other Side) from the same Medium. And therefore the Motion of the Ray can only be affected with the Force of that Medium it at last moves in.

S C H O L I U M.

225. Though it is not known to what precise Distance the retractive Power of any Medium reaches ; yet we are sure it is contained in an exceeding small Compass. And therefore the Curve *HPI*, and consequently the Points *H*, *I* may be taken in Practice only as one Point.

There are few or no Examples among all the Phænomena of Nature that afford so clear a Proof of the prodigious Forces of the small Particles of Matter, as the Motion and Refraction of Light does. For notwithstanding the amazing Velocity of the Rays, and the extremely small Space and Time that any refracting Surface has to act in ; and yet to produce such a sensible Refraction as we see it does, must evince that the Forces exerted on these small Bodies must be surprisingly great, and do really exceed all Comprehension.

P R O B. XIII.

The Velocity of a Globe, moving in a right Line, and its Density, and the Density of the resisting Medium in which it moves being given; to find the Time, Velocity, and Space described. :

Here we suppose the Medium to be uniform, and that the Projectile is acted on by no Force but the Resistance of the Medium, and that to be as the Square of the Velocity.

Let W = Weight of the Globe, d = the Diameter, q = it's Density, p = Density of the Medium, s = a Space of $16\frac{1}{2}$ Feet, b = Space the Globe at first can describe in 1 Second, or the first Velocity; x = any Space described, t = Time of describing it, and v = Velocity at the End of that Time. Here I measure the Velocity by the Space described in a Second, and the Time is Seconds.

1. It is proved by Experiments that the Resistance of the Globe is to the Force by which its Motion may be generated in the Time of describing $\frac{8}{3}$ its Diameter, as the Density of the Fluid to the Density of the Globe nearly. The Velocity generated in a given Body is as the Force and Time conjunctly, therefore the Force is as the Velocity divided by the Times or by Spaces uniformly described in these Times, therefore

fore $\frac{2s}{b} : W :: \frac{b}{\frac{8}{3}d} : \frac{3bbW}{16ds} =$ Force that will generate the Globes Motion in the Time it describes

$\frac{8}{3}d$; therefore $\frac{3bbp}{16dsq} W =$ Resistance of the Globe

C c c

with

F I G.

with Velocity b , and likewise $\frac{3pvv}{16dsq}$ $W =$ Resistance with the Velocity v . Now 1 Second : $2s$ (Velocity of a falling Body) : : $t : 2st =$ Velocity generated in the Time t by Gravity. Now since the Moment of Velocity is, in all Cases, as the Force and Moment of Time ; therefore $2st : Wt : : -\dot{v} : \frac{3pvv}{16dsq} W \times t$; therefore $\dot{v} = \frac{-3pv^2t}{8dq}$, or $v^{-2}\dot{v} = \frac{-3pt}{8dq}$; whence the Fluent $v^{-1} = \frac{3pt}{8dq}$. But when $t = 0$, $v = b$, therefore the Fluent corrected is $\frac{1}{v} - \frac{1}{b} = \frac{3pt}{8dq}$, which reduced is $8dqv + 3pbtv = 8dqb$.

2. 1 Second : $2s$ (Space uniformly described with Velocity $2s$) : : $t : 2st =$ Moment of Space described with Velocity $2s$. And since Moment of Velocity $\propto$ Force $\frac{\text{Force}}{\text{Velocity}} \times$ Moment of Space ; and the Moment of Velocity in falling Bodies was found before $= 2st$.

Therefore $2st : \frac{W \times 2st}{2s} : : -\dot{v} : \frac{3pvvW}{16dsq} \times \frac{x}{v}$;

whence $-\dot{v} = \frac{3pvx}{8dq}$, and $\frac{-\dot{v}}{v} = \frac{3px}{8dq}$. There-

fore the Fluent is $\frac{3px}{8dq} = -2.3025 \times \text{Log. } v$. And

by Correction $\frac{3px}{8dq} = 2.302585 \text{ Log. } \frac{b}{v}$:

COR. 1. Hence $x = \frac{8dq}{3p} \times 2.302585 \text{ Log. } 1 + \frac{3pbt}{8dq}$:
this appears by expunging v .

COR. 2.

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FIG.

COR. 2. If $T = \frac{8dq}{3bp}$, then $v = \frac{Tb}{T+t}$; and $x = bT \times 2.302585 \text{ Log} : \frac{T+t}{T}$; and $\text{Log} : \frac{v}{b} = \frac{-x}{2.302585bT}$.

COR. 3. If the Resistance was to be as any Power of the Velocity v^n ; then by the same way of Reasoning, we should get $\frac{3pv^nW}{16dsqb^{n-2}} = \text{Resistance of the Globe with Velocity } v$; and thence $\frac{-\dot{v}}{v^n} = \frac{3pt}{8dqb^{n-2}}$; and the correct Fluent, $\frac{b^{1-n} - v^{1-n}}{1-n} = \frac{3pt}{8dqb^{n-2}}$: But when $n=1$, $\text{Log} : \frac{b}{v} = \frac{3pbt}{8dq}$.

Likewise $\frac{-\dot{v}}{v^{n-1}} = \frac{3p\dot{x}}{8dqb^{n-2}}$, and the correct Fluent $\frac{b^{2-n} - v^{2-n}}{2-n} = \frac{3px}{8dqb^{n-2}}$.

COR. 4. If it was required to find the Motion of any Sort of Body B moving in a right Line, and resisted as the n^{th} Power of the Velocity; it's done in the same Manner. Let the Resistance of B moving with Velocity 1, be = Weight of the Body w . Then $1 : w :: v^n : wv^n = \text{the Resistance it meets, with Velocity } v$.

Then (see Prob. I.) $-v\dot{v} \propto \frac{Fs}{b}$, and in the Case of falling Bodies, $v=2b$, and $\dot{v}=s$, $F=w=b$. And in the Case of Resistance $F=wv^n$, $b=B$. Therefore $2bs : \frac{ws}{w} :: -v\dot{v} : \frac{wv^n s}{B}$. Therefore $-v\dot{v} = \frac{2bws}{B}$.

FIG. $\frac{2bvw^n s}{B}$, or $-v^{1-n} \dot{v} = \frac{2bws}{B}$; and the Fluent is

$$\frac{b^{2-n} - v^{2-n}}{2-n} = \frac{2bws}{B}.$$

Also $-v \propto \frac{Ft}{b}$, and in the Case of falling Bodies

$\dot{v} = 2bt$, whence $2bt : \frac{wt}{w} :: -v : \frac{wv^n s}{B}$. And

$v^{-n} \dot{v} = \frac{2bw}{B} t$, and the Fluent is $\frac{b^{1-n} - v^{1-n}}{1-n}$

$$= \frac{2bw}{B} t.$$

And hence when $v=0$, or all the Motion is destroyed, and n less than 2, then is $\frac{b^{2-n}}{2-n} = \frac{2bws}{B}$, or if n be less than 1, $\frac{b^{1-n}}{1-n} = \frac{2bw}{B} t$; thence the whole Space and Time s and t will be known. But in all other Cases s and t will be infinite, and n being less than 2, the whole Space $s \propto b^{2-n}$, for any the same Value of n .

SCHOLIUM.

It is here laid down as a Principle that the Resistance of a Globe moving in a resisting Medium, is to the Force by which its Motion may be generated in the Time of describing $\frac{3}{4}$ its Diameter; as the Density of the Medium to the Density of the Globe: Yet I have found by some Experiments that in swift Motions, the Resistance has been greater sometimes by a third or fourth Part. These Experiments I tried in a River, with a Globe of equal Density with the Water, by fastning a Thread to the Globe and to an Index inclosed in a Tube with a spiral Spring: For by the Divisions of the Index, as it was drawn out more or less, I could measure the Resistance.

The

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The Reason of this greater Resistance seems to be, F I G.
 that the Globe being near the Surface of the Water,
 and the Motion of the Water being very swift, the
 Water had not Liberty to diverge upwards from the
 Globe, and upon this Account made the Resistance
 greater than if the Globe had been deeply immersed;
 for then the Water could have diverged freely from
 the Globe in all Directions, and on all Sides.

P R O B. XIV.

*If a Body in a uniform Medium, being uniformly acted
 on by the Force of Gravity, ascends or descends in a
 right Line; To find the Times, Velocities, and Spaces
 described.*

Suppose as before W = *Weight of the Globe.*

d = *its Diameter.*

q = *its Density.*

p = *Density of the Medium.*

$s = 16\frac{1}{2}$ *Feet the Space through which a
 heavy Body descends by Gravity in a Se-
 cond.*

x = *Space described from the beginning of the
 Motion.*

t = *Time of describing* x .

v = *Velocity at the End of the Time* t .

b = *Velocity the Body is projected upwards or
 downwards with. Here I measure the
 Velocity by the Space uniformly described
 in a Second.*

The comparative Weight of the Globe in the Me-
 dium will be $\frac{q-p}{q} W$. And we shall find, as in the

last

FIG.

last Problem $\frac{3pvv}{16dsq} W =$ Resistance of the Globe

moving with Velocity v ; and consequently $\frac{q-p}{q} W$

$\pm \frac{3pvv}{16dsq} W$ is the Force acting on the Globe according as it ascends or descends; call this Force y . Now

$2st' =$ Moment of Velocity generated by Gravity in

the Time t' : And the Moment of Velocity being as the Force and Moment of Time universally; there-

fore $2st' : Wt' :: \mp v' : yt' :: \mp \dot{v} : yt$; whence $t =$

$$\mp \frac{W\dot{v}}{2sy} = \frac{\mp 8dq\dot{v}}{q-p \times 16ds \pm 3pvv}.$$

Again, let a falling Body describe any small Space

z' at the End of 1 Second; then it will be, $2s$ (Space uniformly described with Velocity $2s$) : 1 Second ::

$z' : \frac{z'}{2s} =$ Time of describing z' with Velocity $2s$. And

since the Moment of Space $\propto$ Velocity $\times$ Moment

of Time universally: therefore $z' : 2s \times \frac{z'}{2s} :: \dot{x} : vt$

$$:: \dot{x} : vt. \text{ Whence } \dot{x} = vt = \frac{\mp 8dq\dot{v}}{q-p \times 16ds \pm 3pvv}.$$

Case 1. When the Body ascends, $t = \frac{-8dq\dot{v}}{q-p \times 16ds + 3pvv}$,

and (by Form 5.) the Fluent $t = \frac{-8dq \times .017453}{\sqrt{q-p \times 48dsp}}$

$\times$ Degrees in the Arch whose Tangent is $\sqrt{\frac{3pvv}{q-p \times 16ds}}$

But when $t = 0$, $v = b$; and the Fluent corrected is

$t =$

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F I G.

$$t = \frac{2dq \times .017453}{\sqrt{q-p} \times 3dsp} \times \text{Degrees in the Difference of}$$

the Arches whose Tangents are $\sqrt{\frac{3pbb}{q-p \times 16ds}}$ and

$$\sqrt{\frac{3pvv}{q-p \times 16ds}}; \text{ therefore when } v=0, \text{ the whole}$$

$$\text{Time of Ascent } t = \frac{2dq \times .017453}{\sqrt{q-p} \times 3dsp} \times \text{Degrees in the}$$

$$\text{Arch whose Tangent is } \frac{b}{4} \sqrt{\frac{3p}{q-p \times ds}}$$

$$\text{Likewise } \dot{x} = \frac{-8dqvv}{q-p \times 16ds + 3pvv}, \text{ and (by Form}$$

$$\text{the 4th) the Fluent } x = \frac{-2.3025 \times 8dq}{6p} \text{ Log: } \frac{q-p}{q-p \times 16ds + 3pvv}$$

$$16ds + 3pvv^2: \text{ And when duly corrected } x = \frac{4dq \times 2.30258}{3p}$$

$$\times \text{Log: } \frac{q-p \times 16ds + 3pbb}{q-p \times 16ds + 3pvv}$$

$$\text{Case 2. When the Globe descends } t = \frac{8dqvv}{q-p \times 16ds - 3pvv},$$

$$\text{and (by Form the 6th) } t = \frac{dq \times 2.302585}{\sqrt{q-p} \times 3dsp} \times \text{Log:}$$

$$\frac{\sqrt{q-p \times 16ds} + \sqrt{3pvv}}{\sqrt{q-p \times 16ds} - \sqrt{3pvv}} - \frac{\sqrt{q-p \times 16ds} + \sqrt{3pbb}}{\sqrt{q-p \times 16ds} - \sqrt{3pbb}};$$

and this last Term vanishes when b is 0, or the Body descends from Rest.

$$\text{Also } \dot{x} = \frac{8dqvv}{q-p \times 16ds - 3pvv}; \text{ whence the}$$

Fluent

FIG.

Fluent $x = \frac{8dq \times 2.3025}{-6p} \text{Log: } \frac{q-p \times 16ds - 3pvv}{q-p \times 16ds - 3pbb}$

and when corrected, $x = \frac{4dq \times 2.302585}{3p} \times \text{Log:}$

$\frac{q-p \times 16ds - 3pbb}{q-p \times 16ds - 3pvv}$: where b is $= 0$, when the

Globe descends from Rest.

COR. 1. The greatest Velocity the Globe can acquire by an infinite Descent is $\sqrt{\frac{q-p}{3p}} \times 16ds$:

For when x or t is infinite the Denominator $q-p \times 16ds - 3pvv = 0$.

COR. 2. Let $G = q\sqrt{\frac{d}{q-p \times 3ps}}$.

$H = 4\sqrt{\frac{q-p \times ds}{3p}}$.

$N = \text{Number of the Log. } \frac{.434294t}{G}$

Then $t = 2.30258G \times \text{Log. } \frac{H+v}{H-v}$ or $N = \frac{H+v}{H-v}$

which reduced gives $v = \frac{N-1}{N+1} H$, when the Globe descends.

COR. 3. $x = 2.302585 \times \frac{4dq}{3p} \times \text{Log. } \frac{HH}{HH-vv}$
 $= 2.302585 \times \frac{4dq}{3p} \times \text{Log. } \frac{N+1^2}{4N}$, when the
 Globe descends.

COR. 4. In like Manner the Velocity and Space may be found from the Time when the Globe ascends.

SCHOLIUM.

If we had supposed the Resistance to be as any Power of the Velocity v^n , then we should find (a)

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F I G.

in Cor. 3. of the last Prob.) $\frac{3pv^n W}{16dsq b^{n-2}} = \text{Resist.}$

ance of the Globe with Velocity v : and consequently

$$i = \frac{+8dqb^{n-2} \dot{v}}{q-p \times 16dsb^{n-2} \pm 3pv^n}; \text{ and } \dot{x} = v\dot{t} = \frac{+8dqb^{n-2} v \dot{v}}{q-p \times 16dsb^{n-2} \pm 3pv^n}.$$

And the Fluents will give t and x .

Note, the Density q must always exceed p , otherwise the Globe will not gravitate; contrary to the Supposition.

P R O B. XV.

To find the Velocity and Resistance of a Globe oscillating in a Cycloid, in a resisting Medium.

218.

Let Ba be the Arch described in one entire Oscillation, C the lowest Point, and CZ half the whole Cycloidal Arch equal to the Length of the Pendulum: Let the Globe descend from B , and put $CZ=a$, $CB=b$, $BD=x$, the rest as in the last Problem.

Then we shall find $\frac{q-p}{q} W = \text{comparative}$

Weight of the Globe in the Medium, and $\frac{3pvv}{16dsq} W$

= Resistance of the Globe moving with the Velocity v , in the Point D ; as in the former Problems. Now it is known that CZ is to CD , as the Weight of the

Globe $\frac{q-p}{q} W$ is to its accelerating Gravity at D ,

which therefore is $\frac{q-p}{q} W \times \frac{b-x}{a}$. Therefore the

D d d

F I G. the whole Force by which the Pendulum is urged in

218. D is $\frac{q-p}{aq}bW - \frac{q-p}{aq}Wx - \frac{3pv^2}{16dsq}W = y.$

And therefore we shall find (the same as in Prob. XIV.)

$$\dot{t} = \frac{\dot{v}}{2sy}W, \text{ and } \dot{x} = (vt =) \frac{v\dot{v}}{2sy}W. \text{ Put } f =$$

$$\frac{q-p}{aq} \times 2s, g = \frac{3p}{8dq}, \text{ and } b = \frac{q-p}{aq} \times \frac{3p}{4dq}W:$$

$$\text{and then } \dot{t} = \frac{\dot{v}}{bf - fx - gvv}, \text{ and } \dot{x} = \frac{v\dot{v}}{bf - fx - gvv}:$$

And the Fluents will give t and x .

1. Since $\dot{x} = \frac{v\dot{v}}{bf - fx - gvv}$, finding the Fluent (by the Help of Form the 4th and Rule 8. Prop. X.)

$$\text{we have } x = \frac{2.3025}{-2g} \text{Log: } bf - fx - gvv + \frac{f}{2g};$$

but when $x=0$, $v=0$; therefore the Fluent corrected

$$\text{is } x = \frac{2.302585}{2g} \times \text{Log: } \frac{bf + \frac{f}{2g}}{bf + \frac{f}{2g} - fx - gvv}.$$

Let n = Number belonging to the Logarithm $\frac{2gx}{2.302585}$,

$$\text{and then } n = \frac{2bg + 1}{2bg + 1 - 2gx - \frac{2gg}{f}vv}, \text{ which}$$

$$\text{reduced gives } vv = : \frac{n-1}{n}b + \frac{n-1}{2ng} - x : \times \frac{f}{g}.$$

2. Let $z = \frac{3pvv}{16dsq}W$ the Resistance in D , then

$$vv = \frac{16dsq}{3pW}z, \text{ and } v\dot{v} = \frac{8dsq\dot{z}}{3pW}; \text{ expunge } v \text{ and}$$

$\dot{v}$ out of the Value of $\dot{x}$, and we shall have $\dot{x} =$

$$\frac{\dot{z}}{bb - bx - 2gz}, \text{ and the Fluent } x = \frac{2.3025}{-2g} \text{Log: } bb -$$

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$bx - 2gz + \frac{b}{2g}$: and when duly corrected $x =$

$$\frac{2.302585}{2g} \times \text{Log: } \frac{b + \frac{1}{2g}}{b + \frac{1}{2g} - x - \frac{2g}{b}z}. \text{ Let}$$

$n =$ Number of the Logarithm $\frac{2gx}{2.30258}$; then

$$\text{will } n = \frac{b + \frac{1}{2g}}{b + \frac{1}{2g} - x - \frac{2g}{b}z}, \text{ which reduced gives}$$

$$z = \frac{n-1}{n} b + \frac{n-1}{2gn} - x : \times \text{ into } \frac{b}{2g}.$$

COR. 1. In the lowest Point C, $n =$ Number belonging to the Logarithm $\frac{3pb}{2.30258 \times 4dq}$. And there the

$$\text{Velocity} = \sqrt{\frac{n-1 \times 4dq}{3p} - b} : \times \text{ into } \frac{f}{ng}, \text{ and the}$$

$$\text{Resistance} = : \frac{n-1 \times 4dq}{3p} - b : \times \text{ into } \frac{q-p}{naq} W.$$

COR. 2. But the Velocity and Resistance are the greatest, when $\dot{z}$ or $bb - bx - 2gz \times \dot{x} = 0$, and thence $z = \frac{b-x}{2g} b = \overline{b-x} \times \frac{q-p}{aq} W.$

COR. 3. And therefore the Velocity and Resistance are the greatest when $x = \frac{2.30258 \times 4dq}{3p} \times \text{Log: } \frac{3pb}{4dq} + 1 :$

$$\text{For then } z = \overline{b-x} \times \frac{b}{2g} = \frac{n-1}{n} \times b + \frac{1}{2g} - x : \times \frac{b}{2g} ; \text{ which reduced gives } n = 2bg + 1, \text{ and Log. } n$$

$$\text{or the Number } \frac{2gx}{2.30258} = \text{Log: of } \overline{2bg + 1}.$$

D d d 2

SCHO-

If the oscillating Body is not a Globe, then the Proportion of its Resistance to that of an equal Globe whose Diameter is d , must either be calculated from Prob. XXII. Sect. II. or found by Experiments; let that be as 1 to m : And then we must take $\frac{g}{m}$ instead of g , or $\frac{p}{mdq}$ instead of $\frac{p}{dq}$ in the foregoing Calculations.

P R O B. XVI.

To find the Density of the Atmosphere at any Height, supposing the Force of Gravity to be as any Power of the Distance from the Earth's Center, and the Density of the Air as the Compression.

Let r = Radius of the Earth.

x = Any Distance from the Center.

d = Density of the Atmosphere at the Earth's Surface.

z = Atmosphere's Density at the Distance x .

n = Exponent of the Law of Gravity.

Since the Density is as the Pressure therefore the Moment of the Density $\propto$ Moment of Pressure, that is $\propto$ Moment of Matter $\times$ Force of Gravity: But Moment of Matter $\propto$ Density $\times$ Moment of Space. Therefore the Moment of Density $\propto$ Density and Moment of Space and Force of Gravity; that is $z \propto zx^n$ universally.

Now it is collected from Experiments, that the Weight of 1 Foot high of Air at the Earth's Surface is

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is to the Weight or Pressure of the Atmosphere, as F I G.
1 to 29725 = p , at a mean Density; therefore, let

x be any small Height, and it is $p : d :: x' : \frac{dx'}{p}$ =
Moment of Density at the Earth's Surface. Whence

from the foregoing general Proportion, $\frac{dx'}{p} : dx'^n$

$:: -z' : z x'^n$: therefore $z = \frac{-x'^n z x'}{p r^n}$, and $\frac{z}{x} =$

$\frac{-x'^n x'}{p r^n}$: And (by Form the 1st and 2d) the Fluent

is 2.3025 Log : $z : = \frac{-x'^{n+1}}{n+1 \times p r^n}$. And duly cor-

rected, $2.302585 \times \text{Log} : \frac{z}{d} : = \frac{r^{n+1} - x'^{n+1}}{n+1 \times p r^n}$;

therefore $z = d \times \text{Number of the Log} : \frac{r^{n+1} - x'^{n+1}}{2.302585 \times n+1 \times p r^n}$

$= d \times \text{Number belonging to the Log} : \frac{r^{n+1} - x'^{n+1}}{68444 \times n+1 \times r^n}$.

COR. 1. If e be any small Height above the Earth's
Surface, then $z = d \times \text{Number of the Log} : \frac{-e}{68444}$

COR. 2. If $n=1$, then $z = d \times \text{Number of the Loga-}$
rithm $\frac{r r - x x}{68444 \times 2 r}$.

COR. 3. If $n=0$, $z = d \times \text{Number of the Logarithm}$
 $\frac{r - x}{68444}$.

COR. 4. If $n=-2$, $z = d \times \text{Number belonging to the}$
Logarithm $\frac{r \times r - x}{68444 x}$. In all which e , r and x are
supposed to be taken in Feet.

P R O B.

P R O B. XVII.

To find the Density of the Atmosphere at any Height; supposing the Force of Gravity to be as any Power of the Distance, and the Compression as any Power of the Density.

Let r = Radius of the Earth.

d = Density at the Earth's Surface.

x = any Distance.

z = Density at the Distance x , from the Center.

p = a Length of 29725 Feet.

n = Index of the Force.

m = Index of the Density.

v = compressing Force at the Distance x .

Now by the Hypothesis $v \propto z^m$, and Force $\propto x^n$;

and $mz^{m-1}z' \propto$ Moment of Pressure, that is as the Moment of Space and Density and Force of Gravity:

that is $z^{m-1}z' \propto x^n z'$, or $z^{m-2}z' \propto x^n x'$, universally.

To find the Moment of Density at the Earth's Surface, we have $\dot{v} \propto mz^{m-1}\dot{z}$; therefore (by Prop. II.)

$$v : \dot{v} :: z^m : mz^{m-1}\dot{z} :: z : m\dot{z}, \text{ therefore } \dot{z} = \frac{zv}{mv};$$

but at the Earth's Surface, $z = d$; and (taking any very small Space s) it will be, $p : v :: s' : \dot{v} :: s : \dot{v}$,

$$\text{and } \frac{\dot{v}}{v} = \frac{s'}{p}; \text{ whence } \dot{z} = \frac{ds}{mp} = \text{Fluxion of}$$

Density at the Earth's Surface: Therefore from the universal Proportion; $d^{m-2} \times \frac{ds'}{mp} : r^n s' :: z^{m-2} z'$:

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$-x^n \dot{x} : : z^{m-2} \dot{z} : -x^n \dot{x}$: Whence $z^{m-2} \dot{z} =$
 $\frac{-d^{m-1} x^n \dot{x}}{m p r^n}$; and the Fluent (by Form the 2d) is

$\frac{z^{m-1}}{m-1} = \frac{-d^{m-1} x^{n+1}}{n+1 \times m p r^n}$: And the Fluent corrected

is $\frac{z^{m-1} - d^{m-1}}{m-1} = \frac{d^{m-1}}{n+1 \times m p r^n} \times r^{n+1} - x^{n+1}$:

And by Reduction $z^{m-1} = d^{m-1} + \frac{m-1 \times r}{n+1 \times m p} d^{m-1}$

$-\frac{m-1 \times x^{n+1}}{n+1 \times m p r^n} d^{m-1}$.

P R O B. XVIII.

To find the Diameters of the Earth.

Suppose the Earth to be in the Form of the Spheroid, $APBQ$; AB the Equinoctial, PQ the Axis; and let its mean Radius $CR=1$, $AC=1+v=a$. And $PC=1-v=e$, let CS be the Conjugate to RC , and RT perpendicular to CS , then by Conics $CS =$

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$\sqrt{aa+ee-1} = 1+vv=c$, and $RT = \frac{ae}{c} = 1-2vv = p$. Here I reject the Powers of v above vv as being very inconsiderable.

In the mean Place R a heavy Body falls about 16,0917 Feet in 1 Second; and the versed Sine of the Arch described by R in 1 Second by the Earth's Revolution is ,04, (if $RC = 21000000$ Feet); also as $1 : \sqrt{\frac{1}{2}} :: ,04 : ,0283 =$ to the centrifugal Force in R , as 16,0917 represents the Force of Gravity; therefore

F I G. therefore $16, 12 = d$, will be the gravitating Force at R if the Earth stands still: And this is nearly the same with that of a Spheroid whose Axis is $2R$, and Radius of the (Base or) greatest Circle $\sqrt{ac}$, which

(by Corol. 2. Prob. XI.) is $2p - \frac{4pp}{3\sqrt{ac}} - \frac{2pp}{5ae\sqrt{ac}}$

$\times ac - pp = \frac{2}{3} + \frac{4}{15}v - \frac{187}{30}v^2$, omitting the given Quantities in that Corollary. Also the Force of the Earth at A is nearly the same as a Spheroid whose Axis is AB and Radius of the Base $\sqrt{ae}$, that is (by

the same Cor.) $2a - \frac{4aa}{3\sqrt{ae}} + \frac{2aa \times aa - ae}{5ae\sqrt{ae}} = \frac{2}{3}$

$+ \frac{2}{15}v + \frac{2}{3}vv$. Likewise the Force of the Earth at P

is $2e - \frac{4ee}{3a} - \frac{2ee \times aa - ee}{5a^3} = \frac{2}{3} + \frac{2}{15}v + 4vv$.

For the centrifugal Force at the Equinoctial, it is as $\sqrt{\frac{1}{2}} : 1 + v :: .04 : .05657 \times 1 + v = b + bv$ (by Substitution) = centrifugal Force at A : Also $\frac{2}{3} + \frac{4}{15}v$

$- \frac{187}{30}vv : d :: \frac{2}{3} + \frac{2}{15}v + \frac{2vv}{5} : 1 - \frac{1}{5}v + 10,03vv$

$\times d$ for the gravitating Force at A , if the Earth stood still: from this take $b + bv$ the centrifugal Force, and we get $1 - \frac{1}{5}v + 10,03vv \times d - b - bv$ for the Force of Gravity at A when the Earth is in Motion.

Let $CD = x$, $CE = y$. Since the Gravity and also the centrifugal Force (which is as the Decrease of Gravity) in A and D , are as a or $1 + v$ to x , therefore the gravitating Force of the Earth in D , when the Earth is in Motion will be $1 - \frac{1}{5}v + 11,23vv \times dx - bx$.

Lastly, $\frac{2}{3} + \frac{4}{15}v - \frac{187}{30}vv : d :: \frac{2}{3} + \frac{2}{15}v + 4vv : 1 + \frac{1}{5}v + 15,27vv \times d$ the Force of Gravity at P . And since the Forces in P and E are as e or $1 - v$ to y , therefore the Force at E is $1 + \frac{1}{5}v + 16,47vv \times dy$.

Now

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Now suppose the Weights of the Columns x and y F I G. to be equal; therefore their Moments or Fluxions multiplied into the gravitating Forces at D and E , will be equal; that is $1 - \frac{2}{3}v + 11.23vv \times dx\dot{x} - bxx$

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$= 1 + \frac{2}{3}v + 16.47vv \times dy\dot{y}$, and taking the Fluents, and dividing by $\frac{1}{2}d$, and putting $1+v$ and $1-v$ for

x and y , there comes out $1 - \frac{2}{3}v + 11.23v^2 - \frac{b}{d} \times$

$\frac{1-v^2}{1+v} = 1 + \frac{2}{3}v + 16.47vv \times \frac{1-v^2}{1+v}$; that is $1 - \frac{b}{d} +$

$\frac{2b}{d} \times v + 9.83 - \frac{b}{d} \times v^2 = 1 - \frac{4}{3}v - 15.07v^2$,

Or $vv + ,0639v = ,000141$, whence $v = ,00213$, and $CA - CP = ,00426$: And therefore if the mean Radius of the Earth be 21000000, then $CA - CP = 89460$ Feet or 17 English Miles nearly: Therefore $AC = 21044730$, and $PC = 20955270$ Feet.

SCHOLIUM.

This Computation supposes the Earth every where of equal Density: But since that is not certainly known, nor with what Force a Spheroid accurately attracts a Body when situated out of the Axis; nor whether the Earth itself is exactly a Spheroid: These Things may render this Solution a little incorrect. If the Earth be more rare towards the Equinoctial than towards the Poles; its Height at the Equinoctial will be increased in that Proportion.

E e e

P R O B.

P R O B. XIX.

FIG. 247. *The Velocity and Direction of a Projectile being given; to find the Path it describes in a resisting Medium.*

Let AC be the Direction of the Projectile, ABF the Curve described; draw AD parallel, and CBD perpendicular to the Horizon, and cO infinitely near CB . Also draw Bm parallel to Cc , and Bn parallel to AD .

Call AC , x ; CB , y ; AB , z ; and let m and n be the Sine and Cosine of DAC ; $b =$ Velocity at A , or the Space described in Time 1; $b =$ Space descended thro' in the Time 1, by the Gravity 1, acquiring the Velocity $2b$, in Vacuo; $c =$ Velocity when the Resistance is equal to the Weight in the Fluid, (see Cor. 1. Prob. 14.); $V =$ Velocity at C in the Line AC ; $v =$ the Velocity at B in the Curve $= \frac{\dot{z}V}{\dot{x}}$, by Resolution of Motion.

Since the Resistance is as the Square of the Velocity, therefore cc (Vel.) : 1 (Ref.) :: $vv : \frac{vv}{cc} =$ Resistance in Direction BO , and $\dot{z} : \dot{x} :: \frac{vv}{cc}$ or $\frac{V^2 \dot{z}^2}{cc \dot{x}^2} : \frac{V^2 \dot{z}}{cc \dot{x}} =$ Resistance in Direction Bm , by the Resolution of Forces.

The Body is drawn from the Tangent at B by its Gravity alone; and $\frac{1}{2}\ddot{y}$ is the Space it descends thro', in the Time of describing $\dot{x}$ with Velocity V ; and twice that Space, or $\ddot{y}$, would be described uniformly in the same Time; and the Velocities being as the Spaces uniformly described; we have, $\dot{x} : V :: \ddot{y} : V\ddot{y}$

$\frac{V\ddot{y}}{\dot{x}}$ = Velocity generated by Gravity in that Time.

Now since Gravity does not affect the Motion along AC, the Retardation therein is wholly owing to the Resistance; therefore $-\dot{V}$ being the Velocity destroyed

by the Resistance alone; and $\frac{V\ddot{y}}{\dot{x}}$ the Velocity generated by the Gravity alone; and the Velocities generated or destroyed being as the Forces, it will be, 1

(Gravity) : $\frac{V\ddot{y}}{\dot{x}}$ (Vel.) : : $\frac{V^2\dot{z}}{cc\dot{x}}$ (Ref.) : $-\dot{V}$ (Vel.);

whence $\frac{-\dot{V}}{V^3} = \frac{\dot{z}\ddot{y}}{cc\dot{x}^2}$.

Put $\dot{y} = s\dot{x}$, supposing $\dot{x}$ given, then $\ddot{y} = \dot{s}\dot{x}$, there-

fore $\frac{-\dot{V}}{V^3} = \frac{\dot{z}\dot{s}}{cc\dot{x}}$. But $\dot{z}^2 = \dot{x}^2 + \dot{y}^2 - 2m\dot{x}\dot{y} = \dot{x}^2$

+ $s^2\dot{x}^2 - 2ms\dot{x}^2$, whence $\frac{-\dot{V}}{V^3} = \frac{\dot{s}}{cc} \sqrt{1-2ms+ss}$;

and (by Form 27.) the Fluent is $\frac{1}{2VV} = \frac{1-m}{2cc} L$

$\times \text{Log: } s-m + \sqrt{1-2ms+ss} : + \frac{s-m}{2cc} \sqrt{1-2ms+ss}$;

and corrected, $\frac{1}{VV} - \frac{1}{bb} = \frac{m}{cc} L \times \text{Log:}$

$\frac{s-m + \sqrt{1-2ms+ss}}{1-m} : + \frac{s-m}{cc} \sqrt{1-2ms+ss} +$

$\frac{m}{cc}$; for in A, $s=0$.

Again, Velocity $\times$ Flux. Velocity $\propto$ Force $\times$ Flux. Space, (see Prob. I.); and if $2b$ represents the Velocity generated by Gravity, in describing any Space S ; then $\dot{s}$ will represent the Fluxion of the Velocity; therefore, $2b \times \dot{s}$ (Vel. $\times$ Fl. Vel.) : $1 \times \dot{s}$ (Grav. $\times$ Space)

:: $-\dot{V} : \frac{VV\dot{z}}{cc\dot{x}} \times \dot{x}$, whence $\dot{z} = \frac{-cc\dot{V}}{2bV}$; and the

correct Fluent, $z = \frac{ccL}{2b} \times \text{Log: } \frac{b}{V}$.

Now since $\frac{-\dot{V}}{V} = \frac{2b\dot{z}}{cc}$, and $\frac{-\dot{V}}{V} = \frac{\dot{z}\ddot{y}}{cc\dot{x}\dot{x}}$, if the latter be divided by the former, we have $\frac{1}{VV} = \frac{\ddot{y}}{2b\dot{x}\dot{x}}$, or $2b\dot{x}^2 = VV\ddot{y}$, in Fluxions, $2V\dot{V}\ddot{y} + VV\ddot{\ddot{y}} = 0$, or $\frac{-\dot{V}}{V} = \frac{\ddot{y}}{2\dot{y}} = \frac{2b\dot{z}}{cc}$, or $cc\ddot{y} = 4b\dot{y}\dot{z}$, an Equation between y and z . Or $cc\ddot{y} = 4b\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2 - 2m\dot{x}\dot{y}}$, an Equation between x and y .

Now since $s = \frac{\dot{y}}{\dot{x}} = \frac{S.OBm}{S.BOm}$, and since the Angle at m is given; therefore if we assume any Angle for B , then $\angle O$ will be given, and consequently s , and from thence V ; and from V , z is known.

$$\begin{aligned} \text{Let } E &= \frac{nnL}{cc} \times \text{Log: } \frac{s-m+\sqrt{1-2ms+ss}}{1-m}; + \\ &\frac{s-m}{cc} \sqrt{1-2ms+ss} + \frac{m}{cc}; \text{ then } \frac{1}{VV} = \frac{1}{bb} + E, \\ \text{and } VV &= \frac{bb}{1+bbE}. \text{ Also } z = \frac{ccL}{2b} \times \text{Log: } \frac{b}{V} = \\ &\frac{ccL}{4b} \times \text{Log: } \frac{bb}{VV} = \frac{ccL}{4b} \times \text{Log: } 1+bbE; \text{ and } \dot{z} \\ &= \frac{bbccE}{4b \times 1+bbE} = \frac{bbs \sqrt{1-2ms+ss}}{2b \times 1+bbE}, \text{ and } \dot{x} = \\ &\frac{\sqrt{1-2ms+ss}}{2b \times 1+bbE}. \text{ Also } \dot{y} = s\dot{x} = \\ &\frac{bbs}{2b \times 1+bbE}. \text{ Likewise } v = \frac{\dot{z}}{\dot{x}} V = V \sqrt{1-2ms+ss} \\ &= b \sqrt{\frac{1-2ms+ss}{1+bbE}}. \end{aligned}$$

Hence

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Hence if you take $s=r+u$, and $\dot{s}=\dot{u}$, and if r be taken successively $\pm$ any given Quantities in Arithmetic Progression, and the Value of E be found in a Series, and so proceed by Rule 5, Prob. X. you will get all the Parts of x and y .

COR. 1. In the Vertex F , $s = m$, and $E = \frac{nnL}{cc}$

$\times \text{Log} : \sqrt{\frac{1+m}{1-m}} : + \frac{m}{cc}$. Therefore put $T = \text{Log} :$

Cotan. of half the $\angle ACD - 10$, as found in the Tables;

and then E or $e = \frac{nnLT+m}{cc}$; and $V = \frac{b}{\sqrt{1+bbe}}$,

$v = \frac{bn}{\sqrt{1+bbe}}$, and $z = \frac{ccL}{2b} \times \text{Log} : \sqrt{1+bbe}$:

For by Trigonometry, if $T = \text{Cotan. } \frac{1}{2}C$, the Cotan. of its double, (that is the Sine of A) will be

$\frac{T-1}{T+1} = m$, reduced $T = \sqrt{\frac{1+m}{1-m}} = \text{Cotan. } \frac{1}{2}C$.

COR. 2. If the Projectile be a Cannon Ball, and if δ be the Diameter, and u the Density of another small light Ball, which projected at the same Elevation, with the

Velocity $\beta = b \sqrt{\frac{u-p \times \delta}{q-p \times d}}$, (see Prob. XIV.) This

Ball will describe a Curve in the Air similar to the other Curve described by the Cannon Ball; and whose Height or Base will be to that of the other respectively; as dq to δu .

To prove this, let lt be the Element of the Curve, lrs the Tangent at l , rt perpendicular to the Horizon. Since this Figure is similar to the correspondent Element of the other Curve, therefore the Lines lr , rs , rt , are in given Ratio's to one another. Now the Line rt ($\dot{y}$) is generated by the Gravity, and sr ($\dot{z}$) destroyed by the Resistance, both in the Time ($\dot{t}$) wherein lr ($\dot{x}$) is described with Velocity v . Therefore $sr \propto$

$\frac{\text{Resistance}}{\text{Matter}} \propto \frac{vvdd}{d^3q} \propto \frac{vv}{dq}$. Also $rt \propto \frac{\text{Gravity}}{\text{Matter}}$

$\propto$

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$$\propto \frac{q-p}{q} \times \frac{W}{W} \propto \frac{q-p}{q}. \text{ And } sr \text{ being in a given}$$

$$\text{Ratio to } rt, \frac{vv}{dq} \propto \frac{q-p}{q}, \text{ or } vv \propto q-p \times d, \text{ there-}$$

fore $vv : VV :: q-p \times d : x-p \times \delta :: bb : \beta\beta$, (V being the Velocity in the other Curve); therefore $\beta = b$

$$\sqrt{\frac{x-p \times \delta}{q-p \times d}}.$$

$$\text{Again, Time (in } lr \text{ or } rt) \propto \sqrt{\frac{rt}{\text{Force of Gravity}}}$$

$$\propto \sqrt{\frac{lr \times q}{q-p}}, \text{ and } lr \propto v \times \text{Time} \propto \sqrt{\frac{lr \times q}{q-p}} \times q-p \times d$$

$\propto \sqrt{lr \times dq}$; whence $lr^2 \propto lr \times dq$, or $lr \propto dq$. But if the *Elementa* of two Curves be similar, the whole Curves will be so, whence $lr \propto$ whole Base of the Curve; therefore the Base is as dq ; whence, 1 Base : 2 Base :: $dq : \delta x$.

COR. 3. If the two Balls projected with the Velocities b and β , were to move without any Resistance in the Medium; they would describe two similar Parabolas, whose Bases or Heights are respectively as those of the other two Curves described in the resisting Medium; and β may be

taken $= b \sqrt{\frac{x-p}{qd}}$, as q is vastly greater than p ; or

even $b \sqrt{\frac{x\delta}{qd}}$, if x very much exceeds p . And if the Gravity of the Balls was not at all diminished in the Fluid; the Velocities would be exactly as $\sqrt{dq}$ and $\sqrt{\delta x}$.

That they would describe Parabolas in a non-resisting Medium, is plain from the Theory of Projectiles; which will be similar, because projected at the same Angle of Elevation. But the horizontal Distance of such a Projection, is as the Velocity Square directly, and the Gravity reciprocally, which in the

two Balls will be as $bb \times \frac{q}{q-p}$ to $\beta\beta \times \frac{x}{x-p}$ or $bb \times$
 $x-p$

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$\frac{x-p \times \delta}{q-p \times d} \times \frac{x}{x-p}$, that is as dq to δx , which was

proved to be the Ratio of the Bases of the other two

Curves. Also it was proved before, that $\frac{vv}{dq} \propto \frac{q-p}{q}$,

but $\frac{q-p}{q}$ represents the relative Gravity in the Fluid ; which if it was = 1, then would $vv \propto dq$.

COR. 4. And hence we have a Method of determining the Curve described by a Cannon Ball in the Air, from the Phenomena. For if a Cannon Ball whose Diameter

is d and specific Gravity q , is projected with Velocity b . Make a small Globe of Cork, or rather of Paper, Bladder, &c. made hollow, and whose Diameter is δ , and specifick Gravity x , and colour it red or black ; if this be projected in the Air, with the Velocity β or

$b\sqrt{\frac{x-p}{qd}}$, and at the same Elevation ; it will exhibit to View, the very Curve described by the Cannon Ball, in Miniature. And if a Leaden Ball be

projected along with it, you will have all the Phænomena of the Cannon Ball in the Air. And observing where the two Globes fall, if the Distances of the horizontal Projections be measured, you will have the Space that the Cannon Ball falls short, by Reason of the Resistance of the Medium. But the Leaden Ball ought rather to be projected with the Velocity

$\beta\sqrt{\frac{x}{x-p}}$ or $b\sqrt{\frac{x\delta}{qd}}$; because the Metal Ball looses no sensible Weight in the Air, and it ought to be projected as far with the Gravity 1, as the Cork Ball

is with the Gravity $\frac{x}{x-p}$, without Resistance. But

if a very slender Body can be made, like an Arrow, something heavier before, and of the same specifick Gravity in the whole, as the Cork Ball ; then this
and

F I G. and the Cork Ball may be projected together, with the same Velocity; and this slender Body will be very little resisted by the Air, as well as the Leaden one, and will show how far the Ball would be projected without Resistance. And the same all holds good if the Balls be projected perpendicularly upwards with the same Velocity; then it will appear how far the Cork Ball comes short of the other. And instead of dq and dx , you may take the Weights divided by dd and dd : for $dq \propto \frac{w}{dd}$, w being the Weight in Vacuo.

S C H O L I U M.

This Problem supposes the Air every where of equal Density; but in a Ball shot from one of our greatest Guns, to the Distance of four or five Miles, the Height at the greatest horizontal Distance, will be about a Mile, and there the Density of the Air is about a sixth Part less than that at the Surface; so that the Ball is less resisted there, and consequently flies further. Likewise the Density of the Air is different at different Times, and consequently the Resistance will be something different upon that Account.

PROB. XX.

If Water or any Fluid ascends and descends alternately FIG. 248.
with a reciprocal Motion in the Legs KL, MN of a
cylindrical Canal or Pipe; to find the Time of one Li-
bration or Reciprocation of the Water.

Let AD be the Level or horizontal Line in which
the Surface of the Water stands when at rest; and
let the Water descend from F in the Leg KL, and
ascend from C in the other Leg; and suppose it to be
in E and G at the same Time. Draw EB, DH per-
pendicular to the horizontal Lines BA, GH, and put
 AF or $CD = b$, AE or $GD = x$, Sine of the Angles L,
 $N = p, q$; $v =$ Velocity in E, $t =$ Time of descending
thro' FE; $l =$ Length of the Canal ALND, $w =$ its
Weight, $s = 16\frac{1}{2}$ Feet: The Force with which the Mo-
tion of the Water is accelerated and retarded alter-
nately, is the perpendicular Height of the Water at
E above G. Then, because $EB = px$, and $DH = qx$,
the Pressure of the two Columns EA, DG is $\frac{p+q}{l}wx$,

the accelerating Force. By Prob. I. $v\dot{v} \propto \frac{F\dot{x}}{b}$, and

in falling Bodies, when $v = 2s$, $\dot{v} = \dot{s}$, whence $2s \times \dot{s} :$

$$\frac{ws}{w} : : v\dot{v} : \frac{p+q}{l}wx \times \frac{-\dot{x}}{w}, \text{ whence } v\dot{v} =$$

$$\frac{-p+q}{l} \times 2sxx\dot{x}, \text{ and } \frac{v^2}{2} = -\frac{p+q}{2l} \times 2sxx, \text{ but in}$$

F. $v = 0$, and $x = b$, therefore the Fluent corrected

$$\text{is } v\dot{v} = \frac{p+q}{l} \times 2s \times \overline{bb - xx}. \text{ Also (Prob. I.) } \dot{t} \propto \frac{\dot{s}}{v}$$

F f f

and

and in falling Bodies $t = \frac{\dot{s}}{2s}$, whence $\frac{\dot{s}}{2s} : \frac{\dot{s}}{2s} ::$

$$t : \frac{\dot{x}}{v}, \text{ therefore } t = \frac{\dot{x}}{\sqrt{\frac{p+q}{l}} \times 2s \sqrt{bb-xx}} =$$

$$\sqrt{\frac{l}{p+q \times 2s}} \times \frac{\dot{x}}{\sqrt{bb-xx}}. \text{ And by Form 10, } t =$$

$$\sqrt{\frac{l}{p+q \times 2s}} \times \text{Arch, whose Radius is 1, and Sine } \frac{x}{b}, \text{ and when } x=b, \text{ that Arch} = \frac{3.1416}{2}; \text{ whence}$$

$$t = \sqrt{\frac{l}{p+q \times 2s}} : \times \frac{3.1416}{2}, \text{ the Time of descend-}$$

ing from F to A , and $2t = 3.1416 \sqrt{\frac{l}{p+q \times 2s}}$,
the Time of one Libration of the Water.

COR. 1. Comparing this Value of t with that in Prob. VII. and Cor. 2. it appears, that the Time of one Libration or Undulation of the Water, is equal to the Time of Oscillation of a Pendulum, whose Length is $\frac{l}{p+q}$; which Length is $= \frac{1}{2}l$, when L and N are right Angles.

COR. 2. All the Times of Libration are equal, whether the Height ascended and descended be greater or less.

COR. 3. It matters not what Figure the Pipe is of below the Level of C , provided the Parts above C be streight wherein the Surface of the Water moves.

COR. 4. The Water in one Libration will descend to I , so that $AI=AF$. For when $v=0$, $bb-xx=0$, and $x=+b$ or $-b$.

SCHOLIUM.

If the Leg MN be wider than KL , and its Section to that of KL as c to 1 , we shall get $vv =$
 $p +$

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FIG.
248.

$$\frac{p + \frac{q}{c}}{l + x - \frac{x}{c}} \times - 2sx\dot{x}. \text{ And if } c \text{ be infinite, (or}$$

which is the same) the End *M* immers'd in stagnant Water, then $v\dot{v} = \frac{-2psx\dot{x}}{l+x}.$

There is a great Analogy between the Libration of the Water in the Canal *KLMN*, and the Waves of the Sea, &c. Therefore if a Pendulum be made whose Length is the Breadth of a Wave from Top to Top; then in the Time that it performs one Oscillation, the Waves will advance forward a Space equal to their Breadth.

For (by Cor. 1.) in the Time that the Pendulum ($\frac{1}{2}l$) vibrates once, the Water in the Canal will make one Libration; and (the Length of the Canal being supposed = $\frac{1}{2}$ the Breadth of a Wave,) in the same Time, the Water in the Hollow, or lowest Place between two Waves, will become the highest, that is, the Wave will move forward half its Breadth. And in two Vibrations (or one Vibration of the Pendulum $2l$) will advance a Space equal to the whole Breadth.

P R O B. XXI.

FIG. 249. Let a Bar of Steel (perfectly elastic) be supported at both Ends; and allowing that 10 Hundred Weight Avoirdupoise hanging in the Middle, will just break the said Bar; it is required to find the Weight of a Globe, falling perpendicular 185 Feet on the Middle of the Bar, to have the same Effect.

Let DBE be the Bar of Steel, supported at the Ends D, E . Now before this Prob. can be solved, something more must be assumed than is here expressed. Suppose then, that the Space BA is known, thro' which the Bar is bent before it breaks; and let the Bar, from the Position DBE , be put into the Position DAE , or very near it, and then the 10 C. Weight laid softly upon A , which just breaks it. Now the Weight falling upon it in the Direction FB , is to bend the Bar just thro' the Space BA , to A , where it breaks.

Let $c =$ the Weight (10 C.) capable to break it when suspended at A .

$w =$ Weight sought, which is to break it by falling.

$a = 16\frac{1}{2}$ Feet.

$d = FB$ the Height that w falls $= 185$ Feet.

$b = BA$, the Space the Bar is bent through when it breaks.

then $2\sqrt{ad} =$ Velocity of w at the Point B , (the Velocity being measured by the Feet described in a Second)

$BC = x, Cc = x'.$

Now by the Principles of Mechanics, when the Bar is bent into the Position DCE , it exerts a Force which

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which is as the Distance BC . Therefore $BA(b) : F I G.$

Force at $A(c) :: BC(x) : \frac{cx}{b} = \text{Force at } C.$ There-

249.

fore the Force that w is acted on at C is $\frac{cx}{b} - w.$

Also by Mechanics, $v\dot{v} \propto \frac{F \times s}{B}$, see Prob. I. In

the Case of falling Bodies, if $2a$ is the Velocity

gained in a Second, $\dot{x}$ is the Velocity gained in moving over $\dot{x}$ at the End of 1 Second. And the Weights

of Bodies being as the Quantities of Matter, therefore (from the general Proportion) $2ax$ (Vel. $\times$ Fl. Vel.)

$:\frac{w\dot{x}}{w} (\frac{\text{Weight} \times \text{Fl. Space}}{\text{Body}}) :: -v\dot{v} : \frac{\frac{cx}{b} - w}{w} \times \dot{x},$

hence $-v\dot{v} = \frac{cx - bw}{bw} \times 2ax$, and the Fluent is

$-\frac{vv}{2} = \frac{\frac{1}{2}cx^2 - bwx}{bw} \times 2a$, but at $B, v = 2\sqrt{ad}$,

and $x = 0$, therefore the Fluent corrected is $4ad - vv$

$= \frac{cx^2 - 2bwx}{bw} \times 2a$; but in $A, v = 0$, and $x = b$,

therefore $4ad = \frac{cb - 2wb}{bw} \times 2a$, or $2d = \frac{c - 2w}{w}b$,

and $2dw = bc - 2bw$, whence $w = \frac{bc}{2d + 2b}$, or w

$= \frac{bc}{2d}$ nearly.

COR. 1. Half the Weight that will break the Bar at

A , will break it when laid upon B . For when $d = 0$,

$w = \frac{bc}{2b} = \frac{1}{2}c.$

COR. 2. The Weight, that by falling a given Height

will break any Bar, is very nearly as the Space (b) thro'

which the Bar will bend before it breaks. Hence the Reason why brittle Bodies break sooner by Percussion, than others of equal Strength.

COR.

FIG. COR. 3. Hence the Weight w is *ceteris paribus*, reciprocally as d the Height fallen from, very near.

S C H O L I U M.

Here no Notice is taken of the Weight of the Bar itself. If instead of descending, the Weight w was supposed to be projected horizontally, with a certain Velocity against the Bar DBE , then the Force

at C would be $\frac{cx}{b}$, and $-vv = \frac{cx}{b} \times \frac{x}{w} \times 2a$,

and $4ad - vv = \frac{cx^2}{bw} \times 2a$. And when $v=0, x=b$,

and thence $4ad = \frac{cb}{w} \times 2a$, and $w = \frac{bc}{2d}$, exactly,

where Gravity has no Concern.

P R O B. XXII.

250. If any Number of Ivory Balls of an Inch Diameter be suspended in a right Line, and the first A let fall from any Height upon the next; to find the Velocity of the Motion propagated thro' them.

Here we suppose that the Balls are perfectly elastic, and placed so as just to touch each other, or very near it. But before this Problem can be solved, such an Experiment as this must be made; black the first Ball over with Ink, and let it fall upon the second in an Arch of a Circle from the Height design'd, or with the given Velocity; this will make a black Spot on the other; take its Diameter, and from thence find the Altitude of the Segment, or the versed Sine of half that Spot. Now the Force by which this Motion is communicated from one Ball

to

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FIG.
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to another, is the Elasticity of the small Segment which is flatten'd or dented in by the Stroke, and acts against the other Ball by Way of a Spring, in like Manner as is described in Ex. 8. Prob. XIII. Sect. I. and this Force is as the Height of the Segment, drove in by the Stroke. Now in two Balls meeting each other, there are two Segments struck in, both which exert their joint Force in separating the Balls again, after meeting.

Let w = Weight of a Ball.

p = the Weight which can make an equal Compression in the Globe, as the Stroke did.

a = BC, which represents the versed Sine of the Segment struck in, C being the Vertex.

d = Diameter of a Globe or Ball as A.

c = Velocity it is projected with at first.

s = $16\frac{1}{2}$ Feet.

x = CD.

v = Velocity in D.

t = Time of describing CD, or the Time in which the Vertex C is pressed in, to D, by the Stroke.

Now it is the same Thing, whether we suppose one Globe, moving with Velocity c , against another at rest, or two Globes moving contrary Ways against one another, each with the Velocity $\frac{1}{2}c$; or whether we suppose one Globe moving against an immoveable Obstacle, with Vel. $\frac{1}{2}c$.

By Mechanics, when the Body is given, $v\dot{v} \propto$ Force $\propto \dot{x}$, (see Prob. I.); but in a falling Body $2s$ is the Velocity, and $\dot{x}$ its Fluxion. And in the Col-

lision of the Balls, the Force at D is $\frac{x}{a}p$; therefore

$2s\dot{x} : w\dot{x} :: -v\dot{v} : \frac{px}{a}\dot{x}$, and $-v\dot{v} = \frac{2psx\dot{x}}{aw}$, and

the Fluent $-v^2 = \frac{2spx^2}{aw}$, but at C, $v = \frac{1}{2}c$, $x = 0$,

corrected

FIG.
250.

corrected $\frac{1}{2}cc - vv = \frac{2spx^2}{aw}$. And at last when C

is in B, $v = 0$, and $x = a$, whence $\frac{1}{2}cc = \frac{2sp}{aw} \times aa$,

or $cc = \frac{8sp}{aw} \times aa$; therefore, since $8s$ and w are

given, and $\frac{p}{a}$ is a given Ratio, $cc \propto aa$, or $\frac{c}{a}$ is a

given Ratio. Also by Reduction $p = \frac{ccw}{8as}$; which

put for p gives $\frac{1}{2}cc - vv = \frac{2sxx}{aw} \times \frac{ccw}{8as} = \frac{ccxx}{4aa}$,

which reduced is $vv = \frac{cc}{4aa} \times aa - xx$.

Again $t \propto \frac{\dot{x}}{v}$, and in falling Bodies $t = \frac{\dot{x}}{2s}$ (see

Prob. I.) therefore $\frac{\dot{x}}{2s}$ (Time) : $\frac{\dot{x}}{2s} (\frac{\text{Space}}{\text{Vel.}}) :: t : \frac{\dot{x}}{v}$,

whence $t = \frac{\dot{x}}{v} = \frac{2a\dot{x}}{c\sqrt{aa-xx}}$, and (by Form 10)

$t = \frac{2a}{c} \times \text{Arch}$, whose Sine is $\frac{x}{a}$, and $rad = 1$.

But when $x = a$, that Arch = 90° ; and $t = \frac{2a}{c} \times$

$\frac{3.1416}{2}$; after this Time, the Globes are relatively

at rest; and after an equal Time the Globes are sepa-

rated again. Therefore $2t$ or $\frac{2a}{c} \times 3.1416 = \text{Time}$

of the Contact of the two first Globes; during this

Time the Motion is propagated the Length of 1 Dia-

meter; whence $(2t) \frac{2a}{c} \times 3.1416 : d :: 1 : \frac{dc}{2a \times 3.1416}$

the Velocity of the Motion in a Second.

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It is here supposed that the Action of the first Ball *F I G.* or Globe *A* upon the Second, is over before it affects the third Ball. But if at the same Time the second also acts upon the third, and the third upon the fourth, &c. then the Motion will be propagated further, but then the Time will be longer, which comes to the same Thing. But the Time of Contact of any two Balls is so very small, that the Action of the first upon the second is over, or very near over, before it reaches the third. If the first Ball acted sensibly upon several at once, it would be reflected, which is contrary to Experience.

COR. 1. Hence the Velocity of the Motion is the same, whatever be the Velocity of the first Globe impinging upon the second.

For *d* is given, and $\frac{c}{a}$ is a given Ratio.

COR. 2. Hence also, *ceteris paribus*, the Velocity of the Pulses is as the Diameters of the Globes.

COR. 3. In Globes of different Matter, the Value of *a* must be found, and then the constant Ratio $\frac{c}{a}$; whence the Velocity of the Pulses is had.

P R O B. XXIII.

To find the Velocity of Sound.

Sound is caused by the vibrating Motion of a tremulous Body, which excites a Motion in the Parts of the Air that lie next, and these excite the like Motions in the Parts that lie next to them, and so on. For the Parts of the sonorous Body alternately going and returning, do in going drive before them those Parts of the Medium that lie nearest, and by that Impulse, compress and condense them, and in return-

G g g

ing

F I G. ing, suffer them again to recede and expand themselves. Therefore the Parts of the Medium are condensed and expanded by Turns at each Vibration; and move to and fro, in like Manner as the Parts of the vibrating Body do; and consequently the Particles of the Medium have the same Motion successively communicated to them, and the same Laws of Motion, that the tremulous Body has. Now it is a known Property of the Motion of a vibrating Body, that the Force it exerts in vibrating, is as the Distance from the middle Point of the Vibration, see Prob. II. and in any elastic Body, the Force is as the Distance it is stretch'd to. Therefore each Particle of the Air is acted on with a Force which is also as its Distance from the middle Point of its Vibration, this being premised,

Let some certain Space BC be the Distance of two succeeding Pulses, Ee the Space thro' which a Particle vibrates; O its middle Point. EG a very small linear Part of the Medium, successively transfer'd into the Places $\epsilon\gamma$, eg , and back again. Then,

Put $BC=b$, EO or $Oe=a$, $EG=p$, t = Time of describing $E\epsilon$, τ = Time of describing $G\gamma$, v = Velocity of E at ϵ , n = Force acting at E when it begins first to move, $O\epsilon=x$, the Distance of ϵ from its Center; and y = Distance of γ from its Center at the same Time; A = height of a uniform Atmosphere, $c=3.1416$, $s=16\frac{1}{2}$ Feet.

By the Law of Vibration $a : n :: x : \frac{nx}{a}$ = Force at ϵ , and by Mechanics (see Prob. I.) $v\dot{v} \propto F\dot{x}$, and in a falling Body $v = 2s$, and $F = p$, therefore $2s\dot{x} : p\dot{x} :: v\dot{v} : \frac{nx}{a} \times -\dot{x}$, and $v\dot{v} = \frac{-2nsx\dot{x}}{ap}$, and $v^2 = \frac{-2nsxx}{ap}$, and corrected $v\dot{v} = \frac{2sn}{ap} \times \frac{aa-xx}{aa}$, and (putting $q = \sqrt{\frac{2sn}{ap}}$) $v = q\sqrt{aa-xx}$.

Again

Again $t \propto \frac{\dot{x}}{v}$, and in falling Bodies, $2st' = \dot{x}$,

therefore $t : \frac{2st'}{2s} :: t : \frac{-\dot{x}}{q\sqrt{aa-xx}} = t$, and (by

Form 10) $t = \frac{\text{Arch}}{qa}$, whose Radius is a , and Sine

x ; and by Correction (when $x=a$, that Arch = a

Quadrant, then) $t = \frac{\text{Arch}}{qa}$, whose Cosine is x , let

z = that Arch, then $t = \frac{z}{qa}$, and for the same Rea-

son $\tau = \frac{\text{Arch}}{qa}$, whose Cosine is y . Therefore $t - \tau$

$= \frac{\text{diff. Arches}}{qa}$, whose Cosines are x and y . Let z'

$= \text{diff. Arches}$, and $x' = \text{diff. Cosines}$; and by the

Nature of the Circle, $z' = \frac{-ax'}{\sqrt{aa-xx}}$, therefore t

$$- \tau = \frac{z'}{qa} = \frac{-x'}{q\sqrt{aa-xx}}.$$

As the Particles E, G are successively agitated with like Motions, the Pulse in passing from B to C , arrives at E before it comes to G ; and the Spaces p, b , passed over by the Pulse, are as the Times, that

is (as has been shewn), as $\frac{z}{qa}$, and $\frac{2ca}{qa}$; therefore

$$\frac{z}{p} = \frac{2ca}{b}. \text{ But } E\epsilon - G\gamma \text{ or } EG - \epsilon\gamma = y - x = x',$$

and $\epsilon\gamma = p - x'$. Now since the elastic Force of the Air is as the Density, and the Density reciprocally as the Space taken up; if $d =$ the mean Density, then

FIG.

251.

$$\frac{1}{p} : d :: \frac{1}{p-x} : \frac{pd}{p-x} \text{ or } d + \frac{dx}{p} = d + \frac{dz}{ap} \times$$

$$\sqrt{aa-xx} = d + \frac{2cd}{b} \sqrt{aa-xx}, \text{ the Density at } \gamma.$$

But the mean elastic Force :: is to p (the Weight of the Lineola p) :: $A : p$; therefore the mean elastic

$$\text{Force} = \frac{pA}{p} = A. \text{ And } d : A :: d + \frac{2cd}{b} \sqrt{aa-xx} :$$

$$A + \frac{2cA}{b} \sqrt{aa-xx} = \text{the elastic Force in the Place}$$

$$\gamma; \text{ and its Fluxion is } \frac{2cA}{b} \times \frac{-xx}{\sqrt{aa-xx}}, \text{ the}$$

Fluxion of the Force at ϵ corresponding to x ; and

$$\frac{2cA}{b} \times \frac{-xx}{\sqrt{aa-xx}} \text{ or } \frac{2cAxz}{ab} \text{ is the Decrease of the}$$

Force corresponding to x , at the same Moment of

$$\text{Time : that is, } \frac{2cAxz}{ab} \text{ or } \frac{4ccAxp}{bb} \text{ is the Excess of}$$

the Force at ϵ above that at γ : and this is the Force by which γ is accelerated, and therefore that Force

$$\text{is as } x; \text{ and when } x=a, \text{ that Force is } \frac{4ccapA}{bb} = n.$$

$$\text{Therefore } t = \frac{z}{aq} = \frac{z}{a} \sqrt{\frac{ap}{2sn}} = \frac{z}{a} \sqrt{\frac{bb}{2s \times 4ccA}}$$

$$= \frac{bz}{2ca\sqrt{2sA}}. \text{ And when } x=a, \text{ then } z = \frac{ca}{2},$$

$$\text{and } t = \frac{cab}{4ca\sqrt{2As}} = \frac{b}{4\sqrt{2As}}, \text{ and } 4t = \frac{b}{\sqrt{2As}}$$

the Time of vibrating from E to e , and back again, or the Time of the Pulse running thro' BC . Therefore,

Sect.

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fore, $\frac{b}{\sqrt{2As}}$ (Time) : b (Space) : : 1 : $\frac{b\sqrt{2As}}{b} =$ FIG.
251.

$\sqrt{2As}$, the Velocity of the Pulses in a Second.

At a mean Gravity of the Air, when the Quick-silver stands at 30 Inches, the specific Gravity of Air to Quicksilver is about as 1 to 11890, therefore the Height of a uniform Atmosphere must then be 29725 Feet = A . Therefore $\sqrt{2As} = \sqrt{32\frac{1}{2} \times 29725} = 978$ Feet, the Velocity per Second.

But the Velocity of sound is really greater; for here is no Allowance made for the Diameters of the Particles of Air and Vapour, which are unknown. For as these solid Particles themselves are incompressible, the Velocity is propagated instantaneously thro' them; and therefore the Space found before should be augmented by a Quantity which is to that Space, as the Diameters of the Particles to their Distances: And so we find by Experience, that Sound moves at the Rate of about 1140 Feet in a Second, at a mean Density of the Air.

COR. 1. Hence the Velocity of Sound is always the same in the same Medium, whether it be more or less intense; for the Velocity is $\sqrt{2As}$, in which a is not concern'd.

COR. 2. Hence also the Distances of the Pulses are known, by dividing 1140 by the Number of double Vibrations made in a Second. Thus by Schol. Prob. II. in a String that makes 300 Vibrations in a Second, $\frac{1140}{300} = 7\frac{2}{3}$ Feet, the Breadth of one Pulse.

COR. 3. The Velocity of Sound (*ceteris paribus*), is the same on the Top of a Mountain, as in a deep Valley.

For if the incumbent Weight of the Atmosphere be increased or decreased in any Ratio; its Density (and the Height of the Mercury) will be increased or decreased in the same Ratio, and the Height A , of a uniform Atmosphere of that Density, will continue the same. And therefore,

COR.

FIG. COR. 4. *The Temper of the Air remaining the same, as to heat and cold; the Velocity of Sound will be the same, how much soever the Air be condensed or rarified.*

S C H O L I U M,

252. To explain more particularly the Motion of the Particles of Air, when condensed and expanded as the Pulses move thro' them; let $B, C, \&c.$ be certain Waves or Pulses of the Air moving from B towards C ; draw the Line KF , and ST parallel to it, and at any Place Q , erect Qs , for the Density of the Medium at Q or r , Qr being the mean Density. And suppose O a fixt Point at the Place F , and p a Particle of the Air exceeding near it; and let the Pulses of Air, move successively thro' the Points $O, p, \&c.$ or rather (for the more easy conceiving it) let O be supposed to move with a uniform Motion along the Line FK , thro' the Waves $C, B, \&c.$ at rest. It is plain the Particle p , going with it, cannot move with a uniform Motion, by Reason of the alternate Contraction and Dilatation of the Air, but will move sometimes to one Side, sometimes to the other Side of O , if Op be of a due Quantity. Now whilst O moves from F towards C , the Medium growing denser, p will approach nearer to O , and at C where the Density is greatest, p and O will coincide; and proceeding forward from C to D , the Density still being greater than the Mean, p will still lose Ground, and at D , the Place of mean Density, p will be the furthest behind O ; and in passing from D to H , into a rarer Medium, the Air expanding towards H , p will move faster than O ; and at H , (the Place of least Density), they will coincide. In proceeding to I , the Medium still continuing rarer, p will still move faster than O ; and at I (where the mean Density is,) p will be then the furthest before O , the same as it was at F . And after this Manner the Points O, p , will go thro' all the Waves or Pulses; the Particle p vibrating first
to

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to one Side of O , and then to the other. And the Phenomena will be the same, if O stands still, and the Waves move. The Points O and p , or the System Op represents OE (in Fig. 251.) Now we may observe that when the System is at r , it is in a denser Part of the Wave than it was at F , where it begun to move, that is (Fig. 251.) the Density at $\epsilon\gamma$ is greater than at EG ; for the same Particles of Air are in EG and $\epsilon\gamma$ at different Times, and in different Parts of the Wave F , and r . But ϵ , γ , being contemporary Positions of the Points, E , G ; the Density at ϵ is greater than at γ , as is plain, by Fig. 252. Therefore the greater Density at ϵ will move $\epsilon\gamma$ towards O .

FIG.
252.

And to construct the Curve of Density FCD ; with the Radius OE or $Oe=a$, describe the Circle EVe , take $Os=x$, and erect the Ordinate $sV=u$. Let Arch $EV=z$, mean Density $=1$, $FI=b$. Then in

FK , take $Fr = \frac{bz}{2ca}$, and at r erect $r\epsilon = \frac{2cu}{b}$, and

the Point ϵ is in the Curve, and $r\epsilon$ is the Excess (or Defect) of Density at r . And for the Nature of the

Curve; put $Fr=X$, $r\epsilon=Y$, then $\frac{b}{2c} = \frac{aX}{z} = \frac{u}{Y}$,

whence $aXY=zu$, a mechanical Curve.

P R O B.

P R O B. XXIV.

FIG. 253. *A Cord or small Rope being fixt at one End, and extended by Means of a Weight at the other End; if the Rope be struck near one End, the Stroke will excite a Motion in it, which will run along the Rope in Form of a Wave. To find the Velocity of the Motion.*

We will suppose the Cord to be exceeding slender, or rather a heavy Line, perfectly flexible; let AB be the Line, and suppose the Point D to be impelled as far as to d by the stroke; draw adb parallel to ADB . As soon as the Point D arrives at d , the Line DE will be found in the Position dSE . For the Force being impressed upon the Particles of the Curve at D , and these put into Motion, they will exert their Force upon the Parts next to them, and they upon the next, and so on; so that the first Motion will be successively communicated to all the Parts of the Cord, and they will all, one after another, be carry'd over to the Line ab ; where all their Motion will be destroyed; and they will remain there, if D remains at d . In the mean Time the Curve Line (or Wave) dSE , will be carried along to gCF , and so forward.

As none of the Parts of the Curve besides those at D , received any stroke; it is plain they are all moved by means of the Cord, communicating D 's Motion to all the rest in order. And since the Force which the Cord exerts on any Particle (by Cor. Ex. 16. Prop. XIII.) is as the Curvature there; therefore the Curve near F is convex towards GF . Like-

wife

wise since the Particles of the Cord, after they have received their Motion, are not carried beyond the Line ab , but stop at it; it is plain that near g , the Curve is convex towards gb . It is also evident that at any two Points (n, o) of the Curve gCF , that are equally distant from the Lines AB, ab ; the Curvature will be the same. For as the Motion of any Particle is generated gradually as it moves from the Line AB ; it must be gradually retarded in approaching ab , so as to lose all its Motion there; therefore the Forces, and consequently the Curvature at o and n , are equal and contrary, and at the Middle C is nothing: And so each Particle is accelerated as far as C , and afterwards retarded. Moreover the Parts of the Rope as they are drawn from F towards C , will be drawn nearer into a right Line as they approach to C ; so the Curvature will continually decrease to C , the Point of contrary Flexure, and therefore the Force will likewise decrease towards C ; and hence the Parts of the Curve, gC, FC are exactly similar and equal. This being premised, the Time is to be found wherein the Wave is carried from G to F , or which is the same, wherein a Particle at G is translated to g .

Let Fk or $kf = a$, $km = x$, $mn = y$, $Fn = z$, $Ck = d$, $r =$ Radius of Curvature in F . $c = 3.1416$, and let the Force at n be as x . By Prob. V. Sect. II. If $\dot{z}$

be given, the Radius of Curvature at n is $\frac{\dot{z}\dot{x}}{-\ddot{y}}$ or $\frac{\dot{z}\dot{x}}{\ddot{y}}$ (because x decreases). Then (by Cor. Ex. 16.

Prop. XIII.) $x : a :: r : \frac{\dot{z}\dot{x}}{\ddot{y}}$, and $ar\ddot{y} = \dot{z}x\dot{x}$, and the Fluent is $ar\dot{y} = \frac{1}{2}\dot{z}xx$, but at F , $\dot{y} = \dot{z}$, and $x = a$, therefore $ar \times \dot{y} - \dot{z} = \frac{1}{2}\dot{z} \times xx - aa$; and $2ar\dot{y} = \frac{2ar + xx - aa \times \dot{z}}{\ddot{y} + \dot{x}\dot{x}}$; and $4ar \times \frac{xx - aa \times \dot{y}}{\ddot{y} + \dot{x}\dot{x}} = \frac{2ar + xx - aa}{\ddot{y} + \dot{x}\dot{x}} \times \dot{x}^2$
H h h $\dot{x}^2$

F I G. $\dot{x}^2 = 4aar\dot{x}$ nearly, because a is extremely small in
 253. respect to r ; therefore $\dot{y} = \frac{x\sqrt{ar}}{\sqrt{aa-xx}}$; and (by

Form 10) $y = \frac{\sqrt{ar}}{a} \times \text{Arch}$, whose Radius is a ,
 and Sine x ; but when $x=a$, $y=0$, that Arch is a
 Quadrant; whence $y = \frac{\sqrt{ar}}{a} \times \text{Arch}$ whose Cosine
 is x , and when x becomes $= 0$, that Arch is a Qua-
 drant $= \frac{ca}{2}$, and $y=d$, therefore $d = \frac{1}{2}c\sqrt{ar}$, whence
 $r = \frac{4dd}{cca}$; therefore the Rad. Curvature in n is $\frac{4dd}{ccx}$.

To find the Time of moving from F to m . Let
 t = the Time, v = Velocity in m , $b = 16\frac{1}{2}$ Feet,
 l = Length of the Cord, w = its Weight, p = its

Tension; then $\frac{wz}{l}$ = Weight of the Particle z . Then

(by Ex. 16. Prop. XIII.) $\frac{4dd}{ccx} : z :: p : \frac{pccxz}{4dd}$ =
 Force at m or n . And by Mechanics (see Prob. I.)

$v\dot{v} \propto F \times \dot{s}$; that is, $2b \times \dot{s} : \frac{wz}{l} \times \dot{s} :: v\dot{v} :$

$\frac{pccxz}{4dd} \times -\dot{x}$, and $v\dot{v} = \frac{-2b\dot{c}clx\dot{x}}{4ddw} = -Bxx$ (by
 Substitution), and $vv = -Bxx$, and by Correction,
 $vv = B \times \frac{aa-xx}{a}$.

Again, $t \propto \frac{\dot{x}}{v}$; or $\frac{\dot{s}}{2b} (t) : \frac{\dot{s}}{2b} (\frac{\dot{x}}{v}) :: t : \frac{-\dot{x}}{v}$;
 or $t = \frac{-\dot{x}}{\sqrt{B}\sqrt{aa-xx}}$. And $t = \frac{1}{\sqrt{B}} \times \frac{\text{Arch}}{a}$,
 whose Radius is a , and Sine x ; and by Correction,

$t =$

$t = \frac{1}{\sqrt{B}} \times \frac{\text{Arch}}{a}$, whose Cosine is x ; and when

$x=0$, that Arch is $= \frac{ca}{2}$; therefore $t = \frac{ca}{2\sqrt{Ba}}$

$= \frac{c}{2\sqrt{B}} = \frac{c}{2} \times \sqrt{\frac{4ddw}{2bpccl}} = d \sqrt{\frac{w}{2bpl}}$, the

Time of a Particle moving thro' Fk , or of the Wave moving from C to k , or thro' $\frac{1}{2}GF$ or d ; whence

$d \sqrt{\frac{w}{2bpl}}$ (Time) : d (Space) :: 1 : $\sqrt{\frac{2bpl}{w}}$, the

Velocity of the Wave in a Second.

And since t and y are each of them as the Arch of the Cosine x ; therefore t is as y ; and y increases uniformly, as it ought; for any Point of the Curve advances forward equal Spaces in equal Times. Therefore the Force is as x , as was supposed; and all the Requisites are truly found, and the Problem rightly solved, and the Curve is such as is here described.

COR. 1. Hence in the same Cord and Tension, the Motion will be equally swift, whether the impressed Force at D be greater or lesser; that is, greater Waves and lesser ones have the same Velocity.

COR. 2. The Pulse is swifter when the Tension is greater, and that as the square Root of the tending Force.

COR. 3. When the Pulse has run from one End to the other, it will be returned back again in an equal Time: And thus it will run back and forward, till the Motion be spent.

For the fixt Point at the End cannot obey the Motion of the Wave; and by its Resistance it is equivalent to a new Impulse.

COR. 4. If the Cord is not perfectly flexible, but rigid and stiff; the Velocity will be greater in the sub-duplicate Ratio of the Degree of Stiffness.

For the greater Stiffness supplies the Place of a greater Tension.

FIG.
253.

COR. 5. If several Strokes be made successively at D; or, which is the same Thing, if the End A be shaken backward and forward; there will be generated so many Waves gradually succeeding one another, as P, Q, R, S, and the Distance of two Waves will be known by dividing $\sqrt{\frac{2b/p}{w}}$, by the Number of Strokes or Shakes in a Second.

COR. 6. The Wave or Pulse will run thro' the Length of any Part of the Cord, in the same Time as that Part would perform one Vibration.

This appears from Prob. II. compared with this.

SCHOLIUM.

And to construct the Curve or Figure of the Wave. With the Radius Fk (or a) describe the Quadrant of a Circle, FTV ; make $km = x$, draw the Ordinate mT ,

and erect the Ordinate $mn = \sqrt{\frac{r}{a}} \times \text{the Arch } FT$,

or, which is the same, $mn = \frac{2d}{ca} \times \text{Arch } FT$. And

hence it appears, that this * *funivolant Curve* is the same as the harmonical Curve, constructed in Schol.

Prob. II. For (Fig. 218.) $AE = \frac{ds}{b}$, and $AZ =$

$\frac{d}{b} \times \text{Quadrant}$, and $AZ - AE$ or $ZE = \frac{d}{b} \times \text{Arch}$,

whose Cosine is s . And both these Curves are of the same Nature as the Curve of Density, constructed in the Scholium of the last Problem; and all of them are related to the Figure of Sines.

* Scœnobatical.

PROB.

P R O B. XXV.

To find the Height of the Tides.

FIG.

254.

Let R = Radius of the Earth's Orbit, or the Distance of the perturbing Body, $\odot C$.

p = periodical Time of the Earth round the Sun, in Seconds.

g = Gravity of any Parcel of Matter, as 1 Foot.

b = 16.1 Feet, the Space descended in 1" by Gravity.

π = 3.1416.

By uniform Motion, $p : 1'' :: 2\pi R : \frac{2\pi R}{p} = A$, the Arch described in 1 Second by the Earth; and $\frac{AA}{2R}$ or $\frac{2\pi\pi R}{pp} =$ versed Sine of that Arch. And (the Forces being as the Spaces descended thereby,

in a given Time; therefore) $b : g :: \frac{2\pi\pi R}{pp} : \frac{2\pi\pi g R}{ppb}$

the centripetal Force of the Sun, or the perturbing Body at C . But by the Theory of Gravity, the Radius R is to $3CD$; as the centripetal Force at C , to the perturbing Force at M or D ; that is, $R : 3CD ::$

$\frac{2\pi\pi g R}{ppb} : \frac{6\pi\pi g}{ppb} \times CD$ or $f \times CD =$ the disturbing

Force of the Sun, &c. at D ; putting $f = \frac{6\pi\pi g}{ppb}$.

Or thus,

Let a = Radius of the Earth.

e = its Density.

b = Semi-diameter of the perturbing Body at $\odot$.

$s =$

The DOCTRINE

$s = \text{Sine of its apparent Semi-diameter.}$

$d = \text{its Density.}$

Then the Forces of Bodies being as the Quantities of Matter divided by the Squares of the Distances; and the Matter, as the Density $\times d$ by the Cube of the Radius, we have $\frac{a^3 e}{aa} : g :: \frac{b^3 d}{RR} : \frac{b^3 dg}{RRae} =$ Force of the perturbing Body at C. And by the Theory of Gravity, $R : \frac{b^3 dg}{RRae} :: 3CD : \frac{3dgs^3}{aeR^3} \times CD$, the perturbing Force at D or M. But by Trigonometry $\frac{b}{R} = s$; therefore $\frac{3dgs^3}{ae} \times CD$ or $f \times CD =$ perturbing Force of the Body at M or D, putting $f = \frac{3dgs^3}{ae}$.

Let $APBQ$ be the Earth, $CA = a$, $CP = b$, $CD = y$, $CE = x$, then the disturbing Force at $D = fy$; and the Force of Gravity at $D = \frac{gy}{a}$, and at $E = \frac{gx}{b}$; therefore the Gravity or Force of D towards $C = \frac{gy}{a} - fy$. Now the whole Columns AC , PC , being of equal Pressure at C ; their Fluxions multiply'd by the gravitating Forces must be equal; that is, $\frac{gyy}{a} - fyy = \frac{gxx}{b}$, and the Fluent $\frac{gyy}{2a} - \frac{fyy}{2} = \frac{gxx}{2b}$, or $\frac{yy}{a} - \frac{f}{g} yy = \frac{xx}{b}$, and when $y = a$, and $x = b$, then $a - \frac{f}{g} aa = b$, and $a - b = \frac{f}{g} aa$.

In Case of the Sun, $a = 21000000$, $p = 31557600$, then $a - b = \frac{f}{g} aa = \frac{6\pi\pi aa}{ppb} = 1.63 \text{ Feet} = 19 \frac{1}{2} \text{ Inches}$ the Height of the Solar Tide.

In Case of the Moon, $s = 8.15^{\frac{1}{2}}$, $d = 11$, $e = 9$, and
 $a - b = \frac{f}{g} aa = \frac{3ds^3a}{e} = 7.28$ Feet, the Height
 of the lunar Tide. And the Height by their joint
 Force, when in Conjunction, is 8.91 Feet.

254.

And the same Forms will serve for finding the
 Height of the Tides on any other Planet, or celestial
 Body; *mutatis mutandis*.

S C H O L I U M.

The Height of the Tides here given is supposed to
 be at such Places where the Sun and Moon are ver-
 tical, and also in the Equinoctial. In Places at a
 Distance from the Equator, the Height will be less,
 as the Latitude is greater; their Height will also be
 less, according to the Sun or Moon's Declination from
 the Equator.

P R O B. XXVI.

To find the Precession of the Equinoxes.

255.

Let $ApEP$ represent the Earth, orthographically
 projected upon the Plain of the solstitial Colure, PCp
 the equinoctial Colure; P, p the Poles, AE the Equi-
 noctial, IK a parallel of Latitude, BCD the Ecliptic;
 let $\odot$ be the Sun at an immense Distance, and vertical
 to D , in the Tropic, draw $QCq \perp BD$.

The Earth being an oblate Spheroid or nearly such,
 the Sphere $PAPe$ is encompassed with a solid Crust,
 this Crust is spread all over the Sphere, and goes
 round the Equator in Manner of a Ring, whose
 Thickness there is Aa or Ee ; in Places at a Distance
 from the Equator, as at I , it is thinner. Now the
 Precession

F I G. Precession of the Equinoxes arises from the Action of the Sun and Moon, upon this Crust or exterior Part of the Earth; the Part *P A p a* being attracted toward them, in Lines as *AG* parallel to *CQ*; and the other Side *PE p e* acted on the contrary Way, or in the Direction *FE*; and both conspiring to turn the Earth about, in the order of the Letters *BPQAD*; while the inner Sphere is acted on all Sides alike. Now the Effect of these Forces upon this Crust, and the Motion communicated thereby to the whole Body of the Earth, is what we are now to enquire after; and we shall first begin with the Sun's Force.

1. Take Arch *XL = Xl*, and draw *LM*, *XY*, *lm* parallel to *DB*; and *XN*, *ln* parallel to *Qq*; and put *p* = periodical Time of the Earth about the Sun; *g* = Gravity of any Parcel of Matter, as 1 Foot; *b* = 16.1 Feet, the Space descended in a Second by Gravity, $\pi = 3.1416$; then, by what was demonstrated in the last Problem, we have the disturbing

Force of the Sun $= f \times LM = \frac{6\pi\pi g}{ppb} \times LM$; also

let the mean Radius of the Earth $= a$, *Aa* = *m*, *IX* = *r*, *CX* = *v*, *S.LDCA* = *s*, *Cof. DCA* = *c*; Arch *XL* or *Xl* = *z*, *Sine XL* or *Xl* = *y*, then will *NX* = *sy*, *NL* = *cy*, *XY* = *sv*, and *CY* = *cv*. Therefore *LM* = *cy* + *sv*, *CM* = *cv* - *sy*, *lm* = *sv* - *cy*, *Cm* = *cv* + *sy*.

Now the Force of two Parcels of Matter at *L* and *l*, to wheel the Earth about the Center *C*, is $= CM \times f \times LM + Cm \times f \times lm$; the former acting by the Power of the Lever *CM*, the other by that of *Cm*: that is, the Force of these two Parcels of Matter is $= \frac{cv - sy}{cy + sv} \times cy + \frac{cv + sy}{sv - cy} \times sv : \times f, = 2csf \times \frac{vv - yy}{vv - yy}$; therefore the Fluxion of the Force of all the Matter in the Circumference *IK* is $= 2csfz \times \frac{vv - yy}{vv - yy}$, this Force is directed from *q* towards *p*, *D*; therefore the Flux. of the Force the contrary Way, or in Direction

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FIG.

255.

rection QAD is $= 2csfz \times \overline{yy-vv} = 2csf \times \frac{vy}{\sqrt{rr-yy}}$
 $- 2csfvvz$; whose Fluent (by Form 10 and 11) is
 $= csrrfz - csrfy\sqrt{rr-yy} - 2csfvvz$, and when $v=r$,
 and doubled, the Force of the Circumference IK is
 $= \pi csfr \times \overline{rr-2vv} = \pi csfr \times \overline{aa-3vv}$. Therefore if
 $v = \frac{1}{3}a$, the Force of the Annulus IK is nothing.

2. By the Nature of the Ellipsis, $a:r::m:\frac{mr}{a}$
 $=$ Height of the Matter at the Circumference IK ;
 therefore the Fluxion of the Force in the Annulus
 IK is $= \pi csfrv \times \overline{aa-3vv} \times \frac{mr}{a} = \frac{\pi csfmv}{a} \times$
 $\overline{aa-3vv} \times \overline{aa-vv} = \frac{\pi csfmv}{a} \times \overline{a^4-4aavv+3v^4}$, and
 the Fluent $= \frac{\pi csmf}{a} \times a^4v - \frac{4aav^3}{3} + \frac{3v^5}{5}$;
 and when $v=a$, and doubled, the Force of the whole
 Crust $= \frac{8\pi csma^4f}{15} = \frac{1}{5} \times \frac{\pi^3 csma^4g}{ppb}$; therefore the
 Force is as the Thickness of the Crust; and this is equal
 to the simple Force $\frac{16\pi^3 csma^3}{5ppb\sqrt{\frac{2}{3}}}g$, acting at the Dis-
 tance $a\sqrt{\frac{2}{3}}$ from the Center C , or by the Power of the
 Leaver $a\sqrt{\frac{2}{3}}$. And this is the Force with which the
 Sun disturbs the Earth, when in the Tropic at D ;
 call this simple Force Fcs .

3. But we must find what this Force will be when
 the Sun is in any other Point of the Ecliptic as at H .
 Draw the great Circle $pHRP$, and making Rad. $= 1$,
 let Arch $CH=z$, Sine $CH=y$, then in the Spherical
 Triangle CHR , right angled at R , Rad. (1) : $S.CH$
 $(y) :: S.C (s) : S.HR=sy$; therefore since F consists
 only of given Quantities, the Force of the Sun at H
 is $= Fsy\sqrt{1-ssyy}$; for it was found in the last Ar-
 ticle to be $= F \times$ by the Rectangle of the Sine and
 I i i Cosine

F I G. Cosine of the Sun's Height above the Plane of the

255. Equinoctial. But this very small Force $Fsy\sqrt{1-ssyy}$, acts altogether in the Plane $PKHp$, therefore we must divide it into two Forces, one acting in the Plane PQA which we want; the other in the Plane PCp , perpendicular to the other; this latter Force is destroyed by an equal and contrary Force, when the Sun is equidistant on the other Side of the Tropic; but the other Force acting always one way, is the only one the Earth is annually affected with. The Tan. $HC =$

$$\frac{y}{\sqrt{1-yy}}, \text{ and by Sph. Trig. } 1:c::T.AC\left(\frac{y}{\sqrt{1-yy}}\right):$$

$$\frac{cy}{\sqrt{1-yy}} = \text{Tan. } RC = t; \text{ and } S.RC = \frac{t}{\sqrt{1+tt}}$$

$$= \frac{cy}{\sqrt{1-yy} \sqrt{1+\frac{ccyy}{1-yy}}} = \frac{cy}{\sqrt{1-yy+ccyy}} =$$

$$\frac{cy}{\sqrt{1-ssyy}}, \text{ then to find the Part of the Force acting}$$

in the Plane PQA , it will be, Rad. $(1): Fsy\sqrt{1-ssyy}$
(whole Force) : : $S.RC\left(\frac{cy}{\sqrt{1-ssyy}}\right): Fcsyy$, the Force

in Direction PQ : Therefore this Force is as the Square of the Sun's Distance from the Equinox. From hence to find the mean annual Force, we must first find the Sum of all the $Fcsyy$ in the Circle, or the Fluent of

$$Fcsyyz = \frac{Fcsyy}{\sqrt{1-yy}}, \text{ and the Fluent (by Form 10,$$

$$11) \text{ is } = \frac{Fcsz}{2} - \frac{Fcsy}{2} \sqrt{1-yy}, \text{ and when } y=1,$$

the whole Fl. $= \frac{Fcs\pi}{4}$, and in the whole Circle $= Fcs\pi$; this divided by the whole Circumference 2π ,

the

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the mean Force is $= \frac{1}{2}Fcs$, which is but half the greatest Force, when the Sun was in the Tropic. FIG. 255.

4. Put n = Time of the Earth's diurnal Rotation, and supposing it of uniform Density, its Solidity is $= \frac{4\pi a^3}{3}$. And by Ex. 6. Prob. XXIII. Sect. II.

the Distance of the Center of Gyration, from the Center C of the Earth is $= a\sqrt{\frac{2}{5}}$, and by the Property of the Center of Gyration, if the whole Matter of the Earth was supposed to be collected into that Point, any Force apply'd to move it about the Center C , would generate the same angular Velocity in it, and in the same Time, as it would do in the Earth itself. And since this Force $\frac{1}{2}Fcs$, acts at the same Distance $a\sqrt{\frac{2}{5}}$, therefore it is the same Thing as if that Force was directly apply'd to the Body to move it: Now to find the Motion generated herein. By Mechanics the Space described or generated in a given Time, is as the Force directly, and Matter

reciprocally; therefore: $\frac{g}{1} : b :: \frac{\frac{1}{2}Fcs}{\frac{4}{3}\pi a^3} : \frac{3Fcsb}{8\pi a^3g}$

$=$ Space described in i , by the Matter of the Earth in the Center of Gyration. And $2\pi a\sqrt{\frac{2}{5}}$ (circumf.)

$: 360^\circ :: \frac{3Fcsb}{8\pi a^3g} : \frac{3Fcsb}{16\pi^2 a^4 g \sqrt{\frac{2}{5}}} \times 360 = \frac{3cs\pi m}{2app} \times 360$,

the Angle described about the Earth's Center, by the Center of Gyration, which is equal to the angular Motion of the Earth in i .

5. In the Equator AE , make $Ab = \frac{360}{n}$, and perpendicular to it make bt = Earth's angular Motion

just found. Now since in i of Time any Point A , is carry'd from A to b by the Earth's Rotation; and in the same Time the Point b of the Equinoctial is moved to t , by the Sun's disturbing Force; its plain,

FIG. any Point *A* will pursue the Tract *Atr*; that is, the
 255. Equinoctial Point will be moved from *C* to *r*, in $\frac{360}{n}$.

To find which, in the Triangle *Abt*, *Ab* or *At* ($\frac{360}{n}$)

$$: \text{Rad. } (1) :: bt \left(\frac{3cs\pi m}{2app} \times 360 \right) : \frac{3cs\pi mn}{2app} = S. L$$

bAt, or Arch *ro*; and in the Triangle *roC*, *S.C* (*s*):

$$S.ro \left(\frac{3cs\pi mn}{2app} \right) :: \text{Rad. } (1) : S.rC = \frac{3c\pi mn}{2app} = \text{Sine}$$

of the Precession of the Equinoxes, by the Sun's

Force, in $\frac{1}{n}$; and $\frac{3c\pi mn}{2ap} = S. \text{Precession in a Year.}$

Therefore the Precession is as the periodic Time of the Earth's Revolution about its Axis.

6. Lastly (by Prop. 37. L. III. *Newton's Principia*) the Sun's Force is to the Moon's, as 1 to 4.4815; and the Sum of both is 5.4815 = *q*, therefore the Sine of the Precession in a Year by both Sun and Moon is

$$= \frac{3c\pi mnq}{2ap}, \text{ where } c = 91706, m = 89460 \text{ Feet, by}$$

Prob. XVII. $n = 86160''$, $a = 21000000$, and $p =$

$$31557600; \text{ whence } \frac{3c\pi mnq}{2ap} = .0002755. \text{ But}$$

.0002909 is the Sine of 1 min. or 60; therefore 2909 :

60 :: 2755 : 56.83, the annual Precession of the Equinoxes.

Which being more than it is found to be by Observation, makes it probable, that the Earth is more dense towards the Center; or perhaps, that the Moon's Force to the Sun's is not rightly adjusted, being only collected from the rising of the Tides. But that the Earth is rarer at the Surface appears from this, that the most Part of the Surface is Water.

COR. 1. *If the Earth had no Rotation about its Axis, The Inclination of the Equator to the Ecliptic would decrease*

crease above two Degrees and a half in one Year: And F I G. consequently in a few Years, the Equinoctial would fall 255. into the Ecliptic.

$$\text{For } 1 : pp :: \frac{3cs\pi m}{2app} \times 360 : \frac{3cs\pi m}{2a} \times 360 =$$

2.681 Degrees in one Year.

COR. 2. The other Force acting in Direction PC perpendicular to PQA, (which was useless in finding the Motion of the Equinoxes) changes its Direction four Times in one periodical Revolution. And this generates a Motion, which compounded with the Earth's diurnal Motion, causes a Vibration or nodding of the Earth's Axis, to and from the Ecliptic, twice in a Year. But because the Moon does not accompany the Sun thro' a Quadrant of the Ecliptic, we cannot consider their joint Force; therefore if we regard only the Force of the Sun (for the Effect of the Moon's Force alone will be far less); then the whole Quantity of this Nutation will not amount to a single Second.

COR. 3. But there is a Nutation in the Earth's Axis, caused by the Moon's being out of the Ecliptic. For in the foregoing Computations, we supposed her always in it; but the Moon's Orbit being inclined to the Ecliptic, in an Angle of about 5 Degrees, she will only be in the Ecliptic when she's in the Nodes; and at all other Times she will have a greater or lesser Force to move the Earth, according as her Declination is greater or lesser. And this Force (as was shewn Art. 4.) is as the Rectangle of the Sine and Cosine of Declination, that is, as the Sine of twice the Declination: And from hence arises a Perturbation of all the former Motions, which will be greatest when she is furthest from the Ecliptic, or 90° from the Node. And as the Nodes make a Revolution in about 18.6 Years, therefore this Force will be directed all manner of Ways during one Period of the Nodes, and will therefore cause the Poles of the Equinoctial

F I G. noſſial to wander about the mean Pole, deſcribing a circular Figure in the Heavens, in the ſaid Period of Time.

Now to compute the Quantity of this *Evagation of the Poles*; the Arch rc or 56.83 was meaſured along the Ecliptic; but $s \times 56.83$, or $22.66 = ro = LA$, which meaſures the Motion of the Pole, and that by the joint Forces of both Sun and Moon. Therefore

$\frac{4 \times 48^{15}}{5 \times 48^{15}} \times ro = 18.52$, the Effect of the Moon's Force alone in a Year; and this Force of the Moon in the Ecliptic is as $S. 47^d = .7313$; and at 5 deg. Lat. when the Nodes are at the Equinoxes, as $S. 57^d = .8386$; the diſturbſing Force then will be $= .8386 - .7313 = .1073$, this is the leaſt Quantity of Force; and the greateſt is as $S. 10^d = .1736$, or thereabouts, when the Nodes are in the Solſtices. And in other Poſitions this perturbating Force is different; but we cannot be far wrong if we take a Mean between the greateſt and leaſt, which is $.1404$; whence this

Proportion, $7313 : 1404 :: 18.52 : 3.55$, the annual Effect of the Moon's diſturbſing Force. And $3.55 \times 18.6 = 66.17$; which, as the Motion is directed all Ways, muſt be laid out in the Circumference of a Circle; and then $\frac{66.17}{3.1416} = 21.06$, the Diameter of

the *Circle of Evagation*; which would appear diſtinct in the Heavens, if the other Forces and Motions were away. But we muſt obſerve, that if the Earth is denſer within, and more rare at the Surface, the Diameter of this Circle will come out leſs than is here determined.

To find the Direction of this Motion. It was ſhewn in Art. 5. that when the Solſtice is at D or B , the Equinoctial Circle is moved by the Forces of the Sun and

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F I G.

255.

and Moon, from the Place AC , to the Place Ar ; consequently the Pole P is moved from P towards C ; and the Case is the same with this disturbing Force of the Moon. For suppose vw to be the Moon's Orbit, w the nonageffimal Degree, or highest Point of the Orbit elevated above the Plane of the Ecliptic DB , towards the Pole P ; then v and w being situated at D and B , the Force of the Moon, being now greater, will also move the Pole P towards C . And for the same Reason, when (by the retrograde Motion of the Nodes) w is got to S , P will be moving towards R ; when w is at C , P moves towards A , so that P always moves the same Way, or in the same Direction that w does. Therefore about π , describe the small Circle $\alpha\beta\gamma\delta\epsilon$; and let α be the Place of the Pole when w is at B : Then whilst w passes successively thro' BS , SC , CH , HD ; the Pole P (moving the same Way) will pass thro' $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\epsilon$; the contemporary Positions of w and P being B and α ; S , β ; C , γ ; H , δ ; D , ϵ , &c. and therefore the Circle described $\alpha\gamma\epsilon$ must of Consequence include the mean Pole π . And from hence it is plain, that the moving Pole is in every Place (as suppose at α) nearer to w the highest Point of the Orbit (at B), than the mean Pole π , by the Radius of the little Circle $\pi\alpha$ or $10\frac{1}{2}$ Seconds.

I need scarce mention that this perturbing Force, and its Effect, is greatest in the Moon's Perigee, being nearly in the reciprocal triplicate Ratio of the Moon's Distance, as is well enough known.

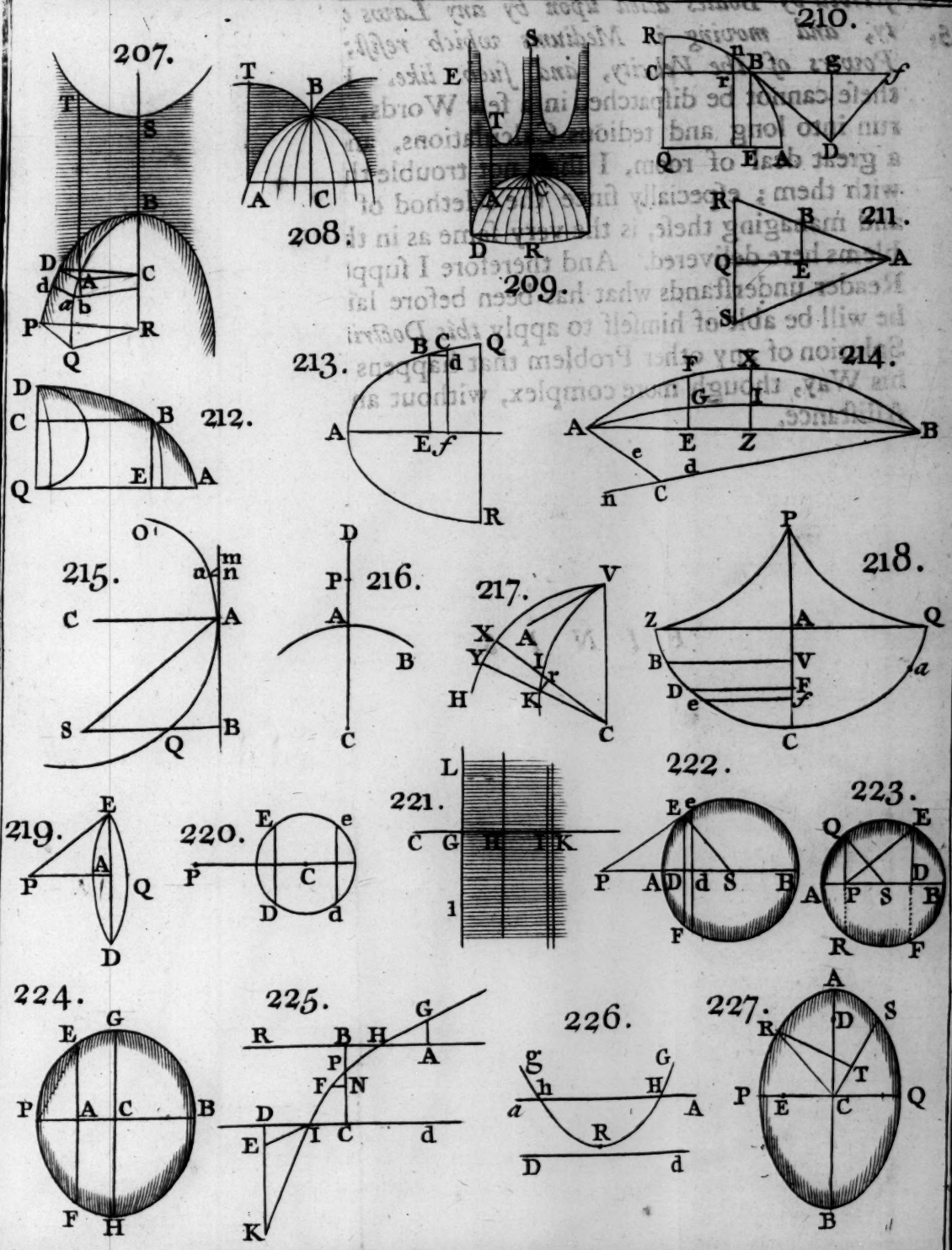
SCHOLIUM.

By a like Method (but by a Process much more easy) the Motion of the Moon's Nodes may be computed. And I might now proceed to the Calculation thereof, and of the Motion of the Apfides, and other Things belonging to the Moon's Motion, and other difficult Problems, such as finding the Curves de-
scribed

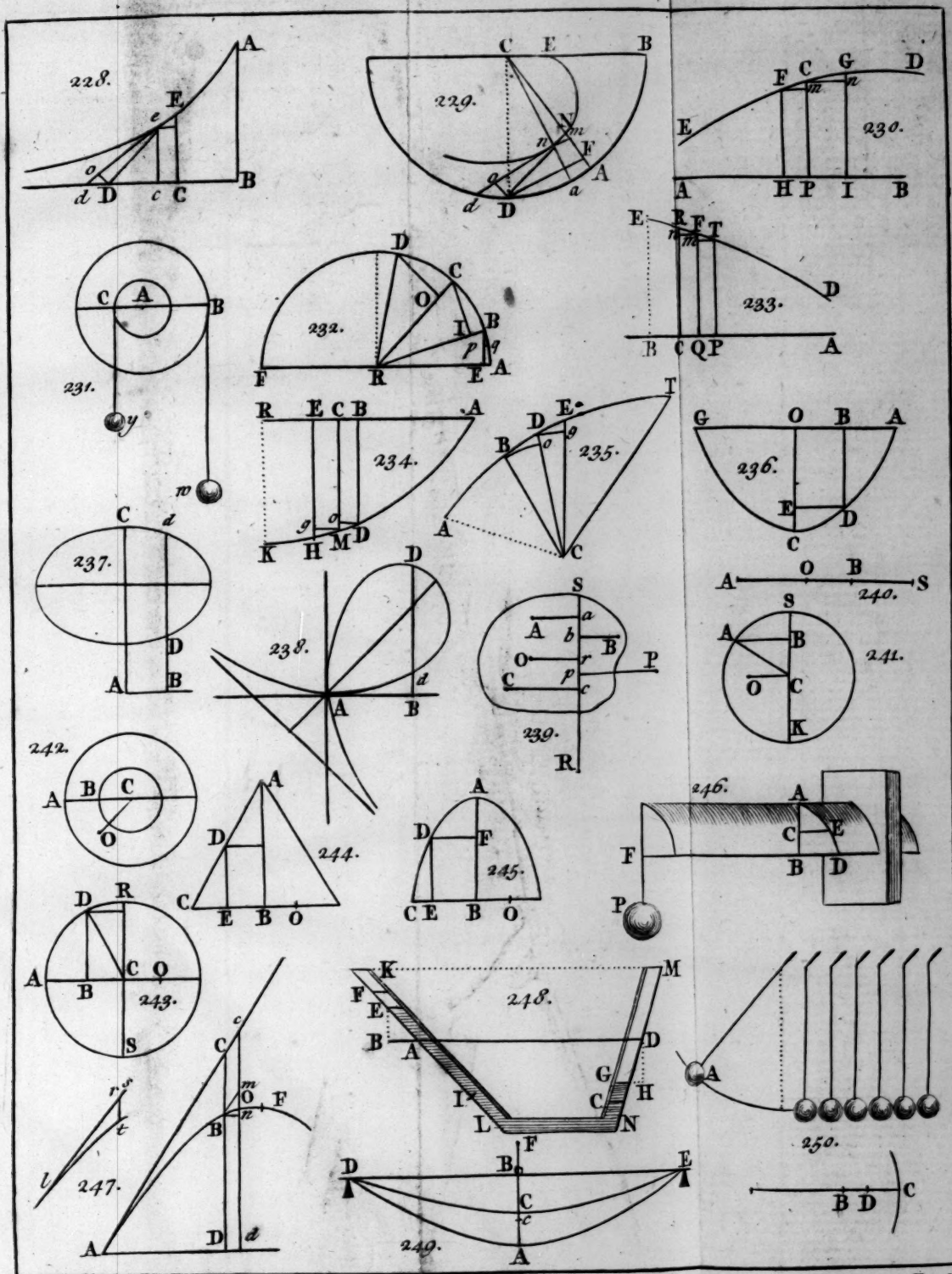
FIG. *scribed by Bodies acted upon by any Laws of Gravity, and moving in Mediums which resist, as any Powers of the Velocity, and such like.* But since these cannot be dispatched in a few Words, but often run into long and tedious Calculations, and require a great deal of room, I shall not trouble the Reader with them; especially since the Method of pursuing and managing these, is the very same as in those Problems here delivered. And therefore I suppose, if the Reader understands what has been before laid down; he will be able of himself to apply *this Doctrine*, to the Solution of any other Problem that happens to fall in his Way, though more complex, without any further Assistance.



F I N I S.









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